Math 306, Spring 2012
Final Exam

Name: ____________________________ Student ID: _________________________

Directions: Check that your test has 11 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). **Please show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 2.5 hours. Good luck!

1. (30 points) ____________
2. (11 points) ____________
3. (7 points) ____________
4. (10 points) ____________
5. (7 points) ____________
6. (12 points) ____________
7. (10 points) ____________
8. (7 points) ____________
9. (7 points) ____________

Total (out of 101): ____________

Curved exam score (out of 100): ____________

Course numerical grade____________ **+ extra credit (out of 100):** ____________

Final course letter grade____________ **Final exam letter grade:** ____________

1. (3 pts each) Give brief answers to the following questions. No explanations are required unless otherwise indicated.

   (a) Give an example of a degree four polynomial over $\mathbb{Q}$ which is irreducible by Eisenstein Criterion.

   (b) Give an example of a field of order 16 which is a quotient of $\mathbb{Z}_2[x]$.

   (c) Is it true that $*$ and $\dagger$ are always inverses of each other? Explain briefly or give a counterexample.

   (d) Give an example of an extension $L\colon K$ such that Gal$(L/K) \cong \mathbb{Z}_3$

   (e) Give an example of a nontrivial Galois extension of $\mathbb{Q}$.

   (f) Give an example of extensions $L\colon M$ and $M\colon K$ which are both Galois, but $L\colon K$ is not Galois.

   (g) Find an irreducible $f \in \mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is nonabelian.

   (h) Find all the subfields of $\mathbb{F}_{625}$. Note that $625 = 5^4$.

   (i) Is it true that the homology groups of any exact sequence are trivial? Explain briefly or give a counterexample.

   (j) Give an example of a functor from the category of sets to the category of groups.

2. (a) (6 pts) Define what it means for a field extension to be (i) simple, (ii) normal, and (iii) separable.

(b) (5 pts) Suppose that $[L : K]$ is a prime number. Prove that $L : K$ is a simple extension.

3. (7 pts) Show that $\mathrm{Gal}(\mathbb{R}/\mathbb{Q})$ is the trivial group.

4. (5 pts each)

(a) If $[K(\alpha) : K]$ and $[K(\beta) : K]$ are relatively prime, show that $[K(\alpha, \beta) : K] = [K(\alpha) : K][K(\beta) : K]$.

(b) Let p be a prime and let a be a rational number which is not a p-th power of another rational number. Let L be the splitting field of $x^p - a \in \mathbb{Q}[x]$. Prove that L is obtained by adjoining a pth real root α of a and a primitive pth root of unity ζ to $\mathbb{Q}$. Also prove that $[L : \mathbb{Q}] = p(p-1)$.

5. (7 pts) Construct the subfield and subgroup lattice diagrams related to the Galois group of $x^4 - 12x^2 + 35 \in \mathbb{Q}[x]$.

6. (12 pts) State all five parts of the Fundamental Theorem of Galois Theory. Prove any three of the five parts, stating carefully any other theorems of lemmas you are using (which you do not need to prove).

7. (a) (5 pts) Show that any finite field must have p^n elements, where p is prime and $n \in \mathbb{N}$.

(b) (5 pts) Show that, given $\mathbb{F}_{p^n}$ and $d \in \mathbb{N}$, there exists an irreducible polynomial $f \in \mathbb{F}_{p^n}[x]$ of degree d.

8. (7 pts) Suppose the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

of abelian groups commutes and the rows are exact. Show that if α and γ surjections, then so is β. (This is a part of the *Short Five Lemma*.)

9. (a) (3 pts) Write down the precise definition of a category.

(b) (4 pts) Show that any group G can be regarded as a category with one object and one morphism for each element of G.