Math 306, Spring 2012
Homework 1, due Friday, February 3

(1) Each of the following statements is false. Disprove each one by providing a counterexample or by appealing to the definition.
  (a) If R is a ring, then for every nonzero $f, g \in R[t]$, we have $\deg(f+g) = \deg(f) + \deg(g)$.
  (b) Every ring R has a field of fractions.
  (c) Every integral domain is a field.
  (d) If K is a field, then $K[t]$ is a field.

(2) Let R be a commutative unital ring. We say that an element $u \in R$ is a *unit* if u has a multiplicative inverse. Prove that the set $U(R) = \{u \in R \colon u \text{ is a unit}\}$ is an abelian group under multiplication.

(3) (a) Using the notation from the previous problem, find the elements of $U(\mathbb{Z}_5)$, $U(\mathbb{Z}_6)$, $U(\mathbb{Z}_{12})$ and $U(\mathbb{Z}_{24})$.
  (b) Use the Fundamental Theorem of Abelian Groups to express each of these groups as a product of cyclic groups of prime power order. No proof required.

(4) Suppose that S is a set and that R is a ring. Let R^S denote the set of all functions $f \colon S \to R$.
  (a) Prove that R^S is a ring, under the operations defined by the following: for all $f, g \in R^S$ and all $s \in S$, let $(f+g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$. You may assume closure and associativity of both operations.
  (b) Prove that, if S has more than one element, then R^S is not an integral domain.

(5) Prove that an integral domain R with a finite number of elements is a field. (Hint: For each nonzero $a \in R$, consider the map $\lambda_a \colon R \to R$ given by $\lambda_a(r) = ar$ for all $r \in R$. Prove that λ_a is injective and use the fact that any injective function on a finite set is surjective.)

(6) Suppose that K is a field. Let $f \in K[x]$, where $f = \sum_{i=0}^{n} a_i x^i$ with $a_i \in K$ for all i. Let $k \in K$. Define $\phi \colon K[x] \to K^K$ by $(\phi(f))(k) = \sum_{i=0}^{n} a_i k^i$.
  (a) Prove that ϕ is a ring homomorphism. (Hint: for multiplicativity, it is best to use summation notation to write $fg = \sum_{i=0}^{n} a_i x^i \sum_{j=0}^{m} b_j x^j = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j} = \sum_{j=0}^{m+n} c_j x^j$ where $c_j = \sum_{i=0}^{j} a_i b_{j-i}$. If this set of equations does not make sense to you, try to verify it by multiplying out some expression like $(a_0 + a_1 x)(b_0 + b_1 x + b_2 x^2)$. To prove additivity, assume without loss of generality that $n \le m$.)
  (b) Prove that, if K is finite, then ϕ is surjective but not injective.

(7) Recall that an ideal I in a commutative unital ring R is *prime* iff $a \in I$ or $b \in I$ whenever $ab \in I$. We say that an element $c \in R$ is *prime* if $c|a$ or $c|b$ whenever $c|ab$. Prove that the following are equivalent for an element $c \in R$:
  (a) the element c is prime in R;
  (b) the ideal (c) is prime in R;
  (c) the quotient $R/(c)$ is an integral domain.