Math 306, Spring 2012
Homework 10, due Monday, April 30

(1) (5 pts) Let p be prime. Construct a tree contaning all the fields $\mathbb{F}_{p^n}$ for $n \in \{1, 2, \ldots, 20\}$ and depicting the subfield structure.

(2) (5 pts) For any prime p, prove that there is an irreducible polynomial $f \in \mathbb{Z}_p[x]$ whose Galois group is $\mathbb{Z}_p$.

(3) (5 pts) Suppose that $f \in \mathbb{Z}[x]$ is an irreducible quartic whose splitting field L has Galois group S_4. Let θ be a root of f and let $M = \mathbb{Q}(\theta)$. Prove that $M\colon \mathbb{Q}$ has degree 4 with no proper subfields. (Hint: Your proof should be by contradiction. You will want to identify the sole subgroup of S_4 with 12 elements, and the 4 subgroups of S_4 with 6 elements.)

(4) (5 pts/part) Let $L\colon K$ be a Galois extension with Galois group G and let $\alpha \in L$. Define the norm and trace of α respectively as

$$\mathsf{N}_{L/K}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha) \quad \text{and} \quad \mathsf{Tr}_{L/K}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

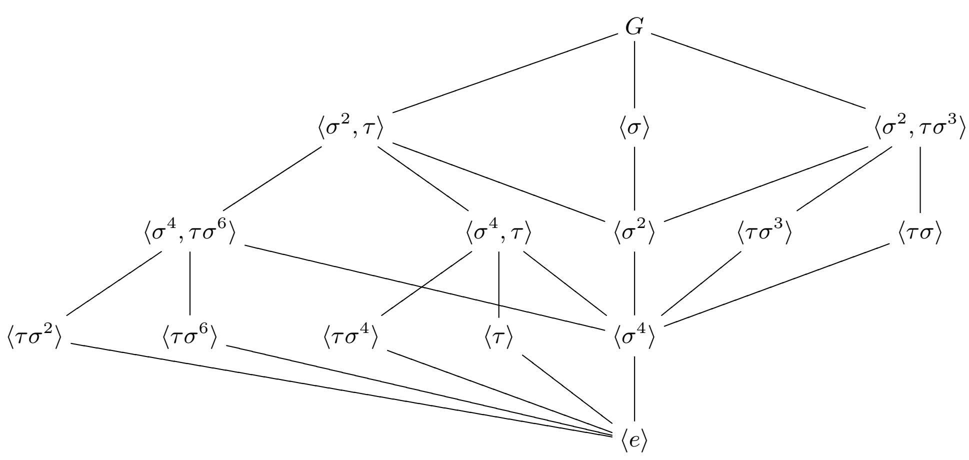
(a) By showing that $\mathsf{N}_{L/K}(\alpha)$ and $\mathsf{Tr}_{L/K}(\alpha)$ are fixed by G, prove that the norm and trace of α are both in K.

(b) Prove that, for all $\alpha, \beta \in L$, we have

$$\mathsf{N}_{L/K}(\alpha\beta) = \mathsf{N}_{L/K}(\alpha)\mathsf{N}_{L/K}(\beta) \;\text{ and }\; \mathsf{Tr}_{L/K}(\alpha + \beta) = \mathsf{Tr}_{L/K}(\alpha) + \mathsf{Tr}_{L/K}(\beta).$$

(c) Let $L = K(\sqrt{D})$ be a quadratic extension of K. Prove that $\mathsf{N}_{L/K}(a + b\sqrt{D}) = a^2 - Db^2$ and $\mathsf{Tr}_{L/K}(a + b\sqrt{D}) = 2a$.

(5) (5 pts) The splitting field of $x^8 - 2$ over $\mathbb{Q}$ is given by $\mathbb{Q}(\sqrt[8]{2}, i)$ which is an extension of degree 16 over $\mathbb{Q}$. If $\zeta = e^{2\pi i/8}$, then every $\mathbb{Q}$-automorphism of $\mathbb{Q}(\sqrt[8]{2}, i)$ is determined by $\sqrt[8]{2} \mapsto \zeta^k \sqrt[8]{2}$ and $i \mapsto \pm i$, where $k \in \{0, 1, \ldots, 7\}$. Let σ be determined by $\sqrt[8]{2} \mapsto \zeta\sqrt[8]{2}$ and $i \mapsto i$ and let τ be determined by $\sqrt[8]{2} \mapsto \sqrt[8]{2}$ and $i \mapsto -i$. The 16-element Galois group G is given by $\langle \sigma, \tau \rangle$, where $\sigma^8 = \tau^2 = e$ and $\sigma\tau = \tau\sigma^3$. Below is a subgroup lattice for G.



The subfields are given by $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i\sqrt[4]{2})$, $\mathbb{Q}(\sqrt{2}\,i)$, $\mathbb{Q}(i)$, $\mathbb{Q}$, $\mathbb{Q}((1+i)\sqrt[4]{2})$, $\mathbb{Q}(i, \sqrt[8]{2})$, $\mathbb{Q}(\zeta^2\sqrt[8]{2})$, $\mathbb{Q}(\zeta^3\sqrt[8]{2})$, $\mathbb{Q}((1-i)\sqrt[4]{2})$, $\mathbb{Q}(i, \sqrt{2})$, $\mathbb{Q}(i, \sqrt[4]{2})$, $\mathbb{Q}(\zeta\sqrt[8]{2})$, $\mathbb{Q}(\sqrt[4]{2})$, $\mathbb{Q}(\sqrt[8]{2})$. Construct the subfield lattice. No explanation required.