Math 306, Spring 2012
Homework 11, due Friday, May 4

Please answer the following questions on the first page of this assignment (this is to determine the amount of extra credit you might receive on the final exam):

(1) How many student seminars did you attend?
(2) Did you give a talk in the student seminar?
(3) Did you attend the colloquium given by Prof. Micheal Ching from Amherst College titled "Apollonian circle packings of the half-plane"?
(4) Did you attend the colloquium given by Prof. Christine Breiner from MIT titled "Existence and uniqueness of minimal surfaces"?
(5) Did you attend the colloquium given by Prof. Elizabeth Denne from Smith College titled "Rectangles and other polygons inscribed in curves"?
(6) Did you attend the colloquium given by Ashley Dombkowski titled "Research, mesearch, wesearch: How consumer-directed generic technologies are changing the face of medicine"?

(1) (5 pts/part)
(a) Show that the sequence of groups

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0,$$

where i and p are the canonical inclusion and projection, is exact.
(b) Prove the *Splitting Lemma*: For a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the following are equivalent:
(i) There exists a map $f' \colon B \to A$ such that $f' \circ f$ is the identity on A.
(ii) There exists a map $g' \colon C \to B$ such that $g \circ g'$ is the identity on C.
(iii) $B \cong A \oplus C$ (so this short exact sequence can be replaced by one from part (a)).
A sequence that satisfies these conditions is called *split exact*.

(2) (5 pts) Suppose $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4$ is exact. Set

$$C_k = \ker(A_k \longrightarrow A_{k+1}) = \operatorname{im}(A_{k-1} \longrightarrow A_k) = \operatorname{coker}(A_{k-2} \longrightarrow A_{k-1})$$

and show that the sequences

$$0 \longrightarrow C_k \xrightarrow{f_k} A_k \xrightarrow{g_k} C_{k+1} \longrightarrow 0$$

are exact. This therefore gives an example of how an exact sequence can be broken into (and spliced from) short exact sequences (the picture of how this works was drawn in class).

(3) (4 pts/part) For this problem, recall that by an extension we mean the entire exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, so that if there are two sequences with the same groups but different homomorphisms between them, we consider those extensions to be different.
(a) Show that any two extensions of C by A have the same number of elements, namely $|C| \cdot |A|$.
(b) Show that there are exactly two non-isomorphic extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$ and that there is exactly one extension of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.
(c) If p is a prime, show that there are exactly p non-isomorphic abelian extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$: the split extension and the extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where p is multiplication by p and i is the multiplication by i for $1 \leq i \leq p-1$.

(4) (5 pts) Show that the sequence

$$\mathbb{Z}/8\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{d_1} \mathbb{Z}/4\mathbb{Z},$$

where each d_i is multiplication by 4, is a chain complex and compute its homology.

(5) (5 pts) Recall the *Short Five Lemma* from lecture: Suppose the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

of abelian groups commutes and the rows are exact. Then
(a) α, γ injections $\Longrightarrow \beta$ is an injection;
(b) α, γ surjections $\Longrightarrow \beta$ is an surjection;
(c) α, γ isomorphism $\Longrightarrow \beta$ is an isomorphism;
Part (a) was proved in lecture. Prove part (b). (Note that part (c) follows from (a) and (b) immediately.)

The remaining problems are optional and can be turned in for extra credit.

(6) (4 pts/part) Show that the following form a category:
(a) A group G (thought of as a category with one object and a morphism for each element);
(b) A poset P (where $X \leq Y$ means that there is a morphism from X to Y);

(7) (4 pts/part) Show that the following maps are functors.
(a) The map $F\colon \mathrm{Sets} \to \mathrm{Sets}$ which sends a set A to its power set $\mathcal{P}(A)$;
(b) The map $F\colon \mathrm{Sets} \to \mathrm{Grps}$ which sends a set A to the free group generated by the elements of A;

(8) (5 pts) For a fixed group H, show that the map $F_H\colon \mathrm{Grps} \to \mathrm{Grps}$ which sends a group G to $G \times H$ is a functor, and that each morphism (homomorphism) of groups $f\colon H \to K$ defines a natural transformation $F_H \to F_K$.

(9) (5 pts) If B and C are groups (regarded as categories with one object each) and $f, g\colon B \to C$ are functors (homomorphisms of groups), show that there is a natural transformation $f \to g$ if and only if f and g are conjugate, i.e. if and only if there is an element $c \in C$ with $g(b) = c(f(b))c^{-1}$ for all $b \in B$.

(10) (5 pts) Given two categories $\mathcal{C}$ and $\mathcal{D}$, one can form a new category $\mathcal{C} \times \mathcal{D}$, called the *product category*. An object of $\mathcal{C} \times \mathcal{D}$ is a pair (c, d) of objects $c \in Ob(\mathcal{C})$ and $d \in Ob(\mathcal{D})$ and a morphism $(c, d) \to (c', d')$ is a pair (f, g) of morphisms $f\colon c \to c'$ and $g\colon d \to d'$. The composition of such morphism is defined componentwise. Show that the product of categories includes the special cases of the product of groups and the product of sets.

(11) (7 pts) Two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are said to be *adjoint* if there is a function

$$\phi\colon \mathrm{Hom}_{\mathcal{D}}(F(-), -) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

which to every pair of objects $c \in \mathcal{C}, d \in \mathcal{D}$ assigns a bijection

$$\phi_{c,d}\colon \mathrm{Hom}_{\mathcal{D}}(F(c), d) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(c, G(d)).$$

Show that the functor $\mathrm{Grps} \to \mathrm{Sets}$ which forgets the group structure and the functor $\mathrm{Sets} \to \mathrm{Grps}$ which to each set assigns the free group on that set are adjoint.