Math 306, Spring 2012
Homework 2, due Friday, February 10

(1) (5 pts) Suppose that R is an integral domain that is not a field. Prove that $R[x]$ is not a principal ideal domain. (Hint: Let $c \in R$ be nonzero and noninvertible and consider $I = (c, x)$.)

(2) (5 pts) Prove that every quotient of a principal ideal domain is also a principal ideal domain. (Hint: Let H be an ideal of R/I and prove that $J = \{a \in R\colon a + I \in H\}$ is an ideal of R and hence principal.)

(3) Let $\mathbb{Q}_{(2)}$ be the set of rationals of the form r/s, where $r \in \mathbb{Z}$ and s is a positive odd integer.
  (a) (3 pts) Prove that $\mathbb{Q}_{(2)}$ is a subring of $\mathbb{Q}$.
  (b) (3 pts) Find the set of units $U(\mathbb{Q}_{(2)})$.
  (c) (5 pts) Prove that $\mathbb{Q}_{(2)}$ is a principal ideal domain. (Hint: Let I be an ideal of $\mathbb{Q}_{(2)}$ and let 2^n be the smallest power of 2 that lies in I; prove that $I = (2^n)$ by showing that $(2^n) \subseteq I$ and $I \subseteq (2^n)$.)

(4) (4 pts/part) For each of the following pairs of polynomials f and g, find the quotient and remainder upon dividing g by f in the polynomial ring $K[t]$, where K is a field.
  (a) $g = t^7 - t^3 + 5$ and $f = t^3 + 1$ in $\mathbb{Q}[t]$;
  (b) $g = t^3 + 2t^2 - t + 1$ and $f = t + 2$ in $\mathbb{Z}_3[t]$.

(5) For each of the pairs above, find the highest common factor and express it in the form $af + bg$ for some $a, b \in K[t]$ (see Theorem 2.9 on page 35). You may leave a and b in unexpanded form. Recall that we defined the highest common factor to be monic.

(6) (a) (5 pts) List all the irreducible monic quadratic polynomials $t^2 + at + b$ in $\mathbb{Z}_5[t]$.
  (b) (3 pts) In each of these cases, compute $a^2 - 4b$. Formulate a conjecture but do not prove it.

(7) (5 pts) List all irreducible cubic polynomials in $\mathbb{Z}_2[t]$.