Math 306, Spring 2012
Homework 3, due Friday, February 17

(1) Each of the following statements is false. Disprove each of them by providing a counterexample or by appealing to the definitions.
  (a) If K is a field, every polynomial in $K[t]$ has a root in K (recall that root is the same as a zero).
  (b) Every polynomial which is irreducible in $\mathbb{Q}[t]$ is also irreducible in $\mathbb{R}[t]$.
  (c) If K is a field and $f, g \in K[t]$ are coprime, then f and g have different degrees.
  (d) If K is a field and $f \in K[t]$ has prime degree, then f is irreducible.
  (e) If K is a field and $f \in K[t]$ has composite degree, then f is reducible.

(2) Determine whether each of the following is reducible or irreducible in the given polynomial ring. If reducible, write it as a product of its irreducible factors. State your reasons briefly.
  (a) $t^3 - 5$ in $\mathbb{Z}_{11}[t]$
  (b) $t^7 + 11t^3 - 33t + 22 \in \mathbb{Q}[t]$
  (c) $t^4 + 1 \in \mathbb{R}[t]$

(3) Prove that the following are all irreducible in the given polynomial ring.
  (a) $x^6 + 539x^5 - 511x + 847$ in $\mathbb{Z}[x]$.
  (b) $x^4 + x^3 + x^2 + 6x + 1$ in $\mathbb{Q}[x]$. (Hint: replace x with $x + 1$.)

(4) Let K be a field and let $f = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree $n \geq 1$.
  (a) Prove that c is a root of f iff $f \in (x - c)$, the ideal generated by $(x - c)$. (Hint: use the Euclidean algorithm on f and $x - c$ and use the fact that $f \in (x - c)$ iff $f = (x - c)g$ for some $g \in K[x]$.)
  (b) Use induction on $n = \deg f$ to prove that f has at most n distinct roots. Please mention unique factorization.