Math 306, Spring 2012
Homework 4, due Friday, February 24

(1) Find the minimal polynomials over the smaller field of the following elements in the given extensions.
- (a) $3i$ in $\mathbb{C}\colon \mathbb{Q}$
- (b) $\sqrt{18}$ in $\mathbb{R}\colon \mathbb{Q}$
- (c) $\sqrt{e}$ in $\mathbb{C}\colon \mathbb{Q}(e)$
- (d) $\frac{\sqrt{5}+1}{2}$ in $\mathbb{C}\colon \mathbb{Q}$
- (e) $e^{2\pi i/11}$ in $\mathbb{C}\colon \mathbb{Q}$

(2) Construct simple extensions $\mathbb{Q}(\alpha)\colon \mathbb{Q}$ where α has the following minimum polynomial in $\mathbb{Q}[t]$.
- (a) $t^2 - 5$
- (b) $t^4 + t^3 + t^2 + t + 1$
- (c) $t^3 + 2$

(3) All the following statements are false. Provide counterexample for each.
- (a) For every finite field K, the polynomial $x^2 + x + 1$ is irreducible in $K[x]$.
- (b) If $K(\alpha)\colon K$ is a simple field extension, then α is algebraic over K.
- (c) There is no nontrivial field extension of $\mathbb{C}(t)$, where t is an indeterminate.
- (d) If K is a simple extension of $\mathbb{Q}$, then there are no subfields properly containing $\mathbb{Q}$ which are properly contained in K.
- (e) If α has minimum polynomial f over $\mathbb{Q}$, then f factors into linear pieces in $\mathbb{Q}(\alpha)$. (Hint: Consider $\alpha = \sqrt[3]{2}$.)

(4) Let p and q be distinct primes in $\mathbb{Z}$.
- (a) Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is a simple extension of $\mathbb{Q}$.
- (b) Prove that $\alpha = \sqrt{p} + \sqrt{q}$ is algebraic over $\mathbb{Q}$ by exhibiting an appropriate quartic polynomial in $\mathbb{Q}[x]$ with α as a root.

(5) Let $\mathbb{F} = \{a + b\sqrt[3]{2} + c\sqrt[3]{4}\colon\ a, b, c \in \mathbb{Q}\}$.
- (a) Prove that $\mathbb{F}$ is closed under multiplication.
- (b) Find the multiplicative inverse of $1 - \sqrt[3]{2}$. (Hint: let the inverse be $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ and solve for a, b, c by multiplying out the appropriate equation.)

(6) Let $\zeta = e^{2\pi i/7}$ be a primitive 7th root of unity. Prove that $\alpha = \zeta + \zeta^{-1}$ is algebraic over $\mathbb{Q}$ by finding a cubic $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. (Hint: Compute α^2, α^3 and α^4 and find a relationship between various powers of α, noticing that $\zeta^3 + \zeta^{-3} = \zeta^4 + \zeta^{-4}$.)

(7) Construct fields of order 9 and 125.