Math 306, Spring 2012
Homework 5, due Friday, March 2

(1) (a) (3 pts) Find a linearly dependent set of three vectors in $\mathbb{R}^3$, but such that any set of two of them is linearly independent.
  (b) (5 pts) Let V be a vector space over $\mathbb{C}$. Suppose that $B = \{v_1, v_2, v_3\}$ is a linearly independent subset of V. Prove that the set $B' = \{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is a linearly independent subset of V.

(2) (5 pts/part) Let K be a field and let K^n be the vector space of n-tuples over K. For all $j \in \{1, \dots, n\}$, let

$$e_j = (0, \dots, 0, 1, 0, \dots, 0, \dots, 0),$$

where the 1 occurs in the j-th position. Let $f_j = e_1 + \cdots + e_j$ for all $j \in \{1, \dots, n\}$.
  (a) Prove that $B_1 = \{e_1, e_2, \dots, e_n\}$ is a basis for K^n.
  (b) Prove that $B_2 = \{f_1, f_2, \dots, f_n\}$ is a basis for K^n.
(Hint: you may assume that the familiar result which says that rows or columns of a square matrix are linearly independent iff matrix is invertible extends to matrices over general fields K.)

(3) (3 pts/part) Find an infinite linearly independent subset of the following vector spaces. No proof is required.
  (a) $\mathbb{R}$ over $\mathbb{Q}$.
  (b) $K(t)$ over K, where K is a field and t is an indeterminate.
  (c) K^S over K, where K is a field and S is an infinite set.

(4) (3 pts/part) Compute the degree $[L\colon K]$ for each of the field extensions below, and exhibit a basis for L as a vector space over K if the degree is finite.
  (a) $\mathbb{Q}(\sqrt[3]{5})\colon \mathbb{Q}$
  (b) $\mathbb{R}(\sqrt[5]{2})\colon \mathbb{R}$
  (c) $\mathbb{Q}(e^{2\pi i/5})\colon \mathbb{Q}$
  (d) $\mathbb{Q}(\sqrt{3}, \sqrt{5})\colon \mathbb{Q}(\sqrt{3})$
  (e) $\mathbb{C}\colon \mathbb{Q}$

(5) Use the Tower Law for the following.
  (a) (5 pts) Prove that, if $L\colon K$ is a field extension with $[L\colon K] = 1$, then $L = K$. (Hint: Show that $K \subseteq L$ and $L \subseteq K$.)
  Remark: We have already used this result in case of simple extensions $K(\alpha)\colon K$ in class. The proof in that case is that if the degree of this extension is 1, then the degree of the monic minimal polynomial over K for α is 1, i.e. the polynomial must be $x - \alpha$, which means that α is in K.
  (b) (3 pts) If $[L\colon K]$ is a prime integer, prove that there are no intermediate fields M strictly between L and K.

(6) (5 pts/part)
  (a) Prove that $B = \{\sqrt{6}, \sqrt{10}\}$ is a linearly independent subset of $\mathbb{R}$ as a vector space over $\mathbb{Q}$.
  (b) Prove that $\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15})\colon \mathbb{Q}$ has degree 4 and not 8. Exhibit a basis for this extension.

(7) (3 pts/part) The following statements are all false. Provide a counterexample or a counterproof. Recall that an extension $L\colon K$ is finite if the degree $[L\colon K]$ is finite.
  (a) Every field extension of $\mathbb{R}$ is a finite extension.
  (b) Every field extension of a finite field is a finite extension.
  (c) There is some element of $\mathbb{C}$ that is transcendental over $\mathbb{R}$.
  (d) If K is a field, then every algebraic extension of K is finite.
  (e) For all $n \in \mathbb{Z}_{\geq 2}$, there are no intermediate fields properly between $\mathbb{Q}(\sqrt[n]{2})$ and $\mathbb{Q}$.

(8) (5 pts/part)

(a) Suppose that $[L\colon K]$ is a prime number. Prove that $L\colon K$ is a simple extension, i.e. there is $\alpha \in L$ such that $L = K(\alpha)$. (Hint: Look at an earlier problem.)

(b) Let $L\colon K$ be a finite extension, and let p be an irreducible polynomial in $K[x]$ with $\deg p \geq 2$. Prove by contradiction that, if $\deg p$ and $[L\colon K]$ are coprime, then p has no zeros in L. (Hint: If $\alpha \in L$ is a root of p, then consider the field $K(\alpha)$.)

(9) (5 pts/part) We say that a rational number a is a *square* in $\mathbb{Q}$ if there is $b \in \mathbb{Q}$ such that $b^2 = a$. Let $m, n \in \mathbb{Q}$ be non-squares. Prove the following.

(a) If mn is a square in $\mathbb{Q}$, then $[\mathbb{Q}(\sqrt{m}, \sqrt{n})\colon \mathbb{Q}] = 2$.

(b) If mn is a non-square in $\mathbb{Q}$, we have $[\mathbb{Q}(\sqrt{m}, \sqrt{n})\colon \mathbb{Q}] = 4$.

(10) (5 pts) Suppose that $M\colon L\colon K$ is a tower of field extensions and let $\alpha \in M$ be algebraic over L. Assume that $[K(\alpha)\colon K]$ and $[L\colon K]$ are relatively prime. Prove that the minimum polynomial m_α^L of α over L actually has its coefficients in K.