Math 306, Spring 2012
Homework 6, due Friday, March 16

(1) (5 pts/part)
   (a) Prove that $\cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$. (Hint: Using the equation $(\cos(2\pi/5) + i\sin(2\pi/5))^5 = 1$, first show that $\alpha = \cos(2\pi/5)$ is a root of $16x^5 - 20x^3 + 5x - 1$, which factors into a linear piece times the square of a quadratic piece.)
   (b) Prove that the regular pentagon is constructible with straightedge and compass.

(2) (5 pts) Prove that the regular 9-gon is not constructible.

(3) (3 pts/part) Find subfields of $\mathbb{C}$ which are splitting fields over $\mathbb{Q}$ for the polynomials (i) $t^3 - 1$, (ii) $t^4 - 1$, (iii) $t^4 - 5t^2 + 6$. Please express your answers without using the letter e.

(4) (2 pts/part) Find the degrees of the field extensions in the previous problem over $\mathbb{Q}$.

(5) (4 pts/part) Determine the splitting field and its degree over $\mathbb{Q}$ for the following polynomials in $\mathbb{Q}[t]$. Here you may use the letter e recklessly.
   (a) $t^4 - 2$
   (b) $t^4 + 2$
   (c) $t^6 - 4$

(6) (4 pts/part) Find a splitting field L for $x^3 - 5$ over (a) $\mathbb{Z}_7$, (b) $\mathbb{Z}_{11}$, (c) $\mathbb{Z}_{13}$. Find the degree $[L\colon \mathbb{Z}_p]$ in each case.

(7) (5 pts/part)
   (a) Let p be prime and let $f = t^p - t + 1$ in $\mathbb{Z}_p[t]$. If α is a root of f, prove that $\mathbb{Z}_p(\alpha)$ is a splitting field for f. (Hint: Prove that $\alpha + 1$ is also a root.)
   (b) Determine the possible values of $[\mathbb{Z}_p(\alpha)\colon \mathbb{Z}_p]$.

(8) (5 pts) Let $f \in K[x]$ have degree n and let L be a splitting field for f over K. Use induction to prove that $[L\colon K]$ divides $n!$. (Hint: break into two cases, where f is irreducible or reducible; you may use the fact that, if $a, b \in \mathbb{Z}_{\geq 0}$, then $a!\,b!$ divides $(a+b)!$.)

(9) (3 pts/part) Decide which of the following extensions are normal. Give reasons for your answer.
   (a) $\mathbb{Q}(t)\colon \mathbb{Q}$, where t is an indeterminate
   (b) $\mathbb{Q}(\sqrt[7]{5})\colon \mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt{5}, \sqrt[7]{5})\colon \mathbb{Q}$

(10) (5 pts) Prove that, if $L\colon K$ is a field extension with $[L\colon K] = 2$, then $L\colon K$ is a normal extension. (Remark: This is analogous to the fact that, if G is a group and H is a subgroup of G with $[G\colon H] = 2$ (the usual index of a subgroup), then H is normal in G.)