Math 306, Spring 2012

Homework 7, due Friday, March 30

(1) We say that a field L is *algebraically closed* if every $f \in L[x]$ splits over L. We know, for example, that $\mathbb{C}$ is algebraically closed. We say that $L\colon K$ is an *algebraic closure* of K if $L\colon K$ is algebraic and L is algebraically closed. Prove that the following are equivalent about an extension $L\colon K$.

- (a) The extension $L\colon K$ is an algebraic closure of K;
- (b) The extension $L\colon K$ is algebraic, and every irreducible $f \in K[x]$ splits over L;
- (c) The extension $L\colon K$ is algebraic, and if $L'\colon L$ is algebraic then $L = L'$.

(2) Construct the normal closures N for the following extensions.

- (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})\colon \mathbb{Q}$
- (b) $\mathbb{Q}(\sqrt[5]{3})\colon \mathbb{Q}$
- (c) $\mathbb{Z}_3(t)\colon \mathbb{Z}_3$, where t is an indeterminate.

(3) For each of these algebraic extensions, find the normal closure M and determine an appropriate collection S for which M is the splitting field over K (this means that each polynomial in the collection splits in M).

- (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots)\colon \mathbb{Q}$
- (b) $\mathbb{Q}(e^{2\pi i/3}, e^{2\pi i/5}, e^{2\pi i/7}, e^{2\pi i/11}, \ldots)\colon \mathbb{Q}$
- (c) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[5]{2}, \ldots)\colon \mathbb{Q}$

(4) Each of the following statements is false. Disprove each of them by providing a counterexample or a counterproof.

- (a) Every finite extension is separable.
- (b) Every normal extension $L\colon K$ is the splitting field of some polynomial $f \in K[x]$.
- (c) For all fields K, if $f \in K[x]$ and $Df = 0$, then $f = 0$.
- (d) Every separable extension is normal.
- (e) Every normal extension is separable.

(5) Suppose that $L\colon K$ is an algebraic extension. Prove that there is a greatest intermediate field M for which $M\colon K$ is normal (assume there is at least one such M). In your proof, you should give a definition of the notion of "greatest".

(6) Let $L\colon K$ be an algebraic field extension and let M_1 and M_2 be intermediate fields normal over K. Define $K(M_1, M_2)$ to be the smallest subfield of L containing both M_1 and M_2. Prove that both $K(M_1, M_2)\colon K$ and $M_1 \cap M_2\colon K$ are normal extensions.

(7) Suppose that f is a polynomial in $K[x]$ of degree n and either $\operatorname{char} K = 0$ or $\operatorname{char} K > n$. Suppose that $\alpha \in K$. Prove that

$$f = f(\alpha) + Df(\alpha)(x-\alpha) + \frac{D^2 f(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{D^n f(\alpha)}{n!}(x-\alpha)^n.$$

(Hint: Proceed by induction on n, using the following fact: If f has degree $k+1$, then α is a root of the polynomial $f - f(\alpha)$, so $f - f(\alpha) = (x-\alpha)g$, for some g of degree k.)

(8) Suppose that f is a polynomial in $K[x]$ of degree n and either $\operatorname{char} K = 0$ or $\operatorname{char} K > n$. Prove that α is a root of multiplicity r iff

$$f(\alpha) = Df(\alpha) = \cdots = D^{r-1} f(\alpha) = 0$$

and $D^r f(\alpha) \neq 0$. (Hint: Proceed by induction on r.)

(9)

- (a) Show that, if $f \in K[x]$ is irreducible and the characteristic of K is p for some prime p, then f is inseparable iff $f = a_0 + a_1^p + \cdots + a_n x^{np}$ for some $n \in \mathbb{Z}_{\geq 1}$ and $a_0, \ldots, a_n \in K$.
- (b) Suppose that $L\colon K$ is a field extension and $\operatorname{char} K = p > 0$. If $[L\colon K]$ is coprime to p, then prove that $L\colon K$ is separable.
- (c) We say that a field K is *perfect* if every irreducible $f \in K[x]$ is separable. Prove that any algebraic extension of a perfect field is also perfect.