Math 306, Spring 2012
Homework 8, due Friday, April 6

(1) Determine the Galois group Gal(L/K) for each of these extensions $L\colon K$. Define the elements as precisely as possible. Here you should not assume that if $L\colon K$ is finite and normal, then the number of elements in Gal(L/K) is $[L\colon K]$.
  (a) $\mathbb{Q}(\sqrt{7})\colon \mathbb{Q}$
  (b) $\mathbb{Q}(\sqrt[5]{2})\colon \mathbb{Q}$
  (c) $\mathbb{Q}(e^{2\pi i/5})\colon \mathbb{Q}$
  (d) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})\colon \mathbb{Q}$
  (e) $\mathbb{Z}_2(\zeta)\colon \mathbb{Z}_2$, where ζ is a root of $x^2 + x + 1 \in \mathbb{Z}_2[x]$

(2) Let $\gamma = \sqrt{2 + \sqrt{2}}$. The purpose of this problem is to compute the Galois group of $\mathbb{Q}(\gamma)\colon \mathbb{Q}$.
  (a) Compute the minimum polynomial $f \in \mathbb{Q}[x]$ of γ. Be sure to verify that f is indeed irreducible. Compute all the roots of f.
  (b) Let β be the other positive root of f. By showing that $\beta = \frac{\sqrt{2}}{\gamma}$, prove that $\mathbb{Q}(\gamma)$ is a splitting field for f over $\mathbb{Q}$.
  (c) By considering the order of the $\mathbb{Q}$-automorphism α satisfying $\alpha(\gamma) = \beta$ (we know there is one by Theorem 7), prove that Gal$(\mathbb{Q}(\gamma)/\mathbb{Q}) \cong \mathbb{Z}_4$.

(3) Given $f \in K[x]$, we say the that Galois group of f is the Galois group of the extension $L : K$ where L is the splitting field of f over K. Consider $f = (x^2 - 2)(x^2 - 3)(x^2 - 5) \in \mathbb{Q}[x]$.
  (a) Determine the Galois group G of f, listing all its elements using the $\mapsto$ notation.
  (b) For each subgroup H of G, compute $H^\dagger$.

(4) Find the Galois group of $x^3 - 5$ over the following fields. List all the subgroups H and the fixed field $H^\dagger$.
  (a) $\mathbb{Q}$
  (b) $\mathbb{Z}_3$
  (c) $\mathbb{Z}_7$