Math 306, Spring 2012
Homework 9, due Friday, April 20

(1) (10 pts) Let p be prime. Prove that the Galois group of $x^p - 2 \in \mathbb{Q}[x]$ is isomorphic to the group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{Z}_p \text{ and } a \neq 0 \right\}.$$

(Hint: Let $\sigma_{k,\ell}$ be determined by $\sqrt[p]{2} \mapsto \sqrt[p]{2}\,\omega^k$ and $\omega \mapsto \omega^\ell$, where $\omega^p = 1$, and prove that $\sigma_{k,\ell} \circ \sigma_{m,n} = \sigma_{k+\ell m,\ell n}$.)

(2) (4 pts/part)
- (a) Let $f \in \mathbb{R}[x]$. Suppose that $z \in \mathbb{C}$ is a root of f. Prove that $\overline{z}$ (complex conjugate of z) is also a root of f.
- (b) Suppose that $f \in \mathbb{Q}[x]$ has degree 3. Use (a) to prove that, if the Galois group of f is isomorphic to $\mathbb{Z}_3$, then f has only real roots. (Hint: Prove the contrapositive, noticing that the conjugation automorphism has order 2.)

(3) (4 pts/part) Consider the polynomial $f = x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$. Recall that we say that an extension $L\colon K$ is *Galois* or that L is *Galois over* K if $L\colon K$ is a normal and separable extension.
- (a) Prove that f is irreducible and find the roots of f.
- (b) Let α and β be roots of f that are not negatives of each other. Prove that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{3})$. (Hint: Consider the degree $[\,\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta)\colon \mathbb{Q}(\sqrt{3})\,]$.)
- (c) Prove that $\mathbb{Q}(\alpha), \mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha, \beta)$ are all Galois over $\mathbb{Q}(\sqrt{3})$.
- (d) Prove that the Galois group of $\mathbb{Q}(\alpha, \beta)\colon \mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- (e) Prove that the Galois group of $\mathbb{Q}(\alpha, \beta)\colon \mathbb{Q}$ is isomorphic to D_8 (recall that this is the dihedral group from 305; if you can't find it in your notes, look it up on Wikipedia). (Hint: Prove first that each σ in the Galois group is determined by its action on α and $\sqrt{2}\,i$.

(4) (3 pts/part) Construct the subfield and subgroup lattice diagrams for extensions related to the following polynomials in $\mathbb{Q}[x]$. You may use any previous results about these polynomials.
- (a) $x^4 - 4x^2 + 2$
- (b) $x^4 - 12x^2 + 35$

(5) (3 pts/part) Use the Fundamental Theorem of Galois Theory to prove the following. Let $L\colon K$ be a Galois extension with Galois group G.
- (a) Suppose that M and N are intermediate fields with $M \subseteq N$. Prove that $N\colon M$ is normal iff N^* is normal in M^*.
- (b) In this case, prove that the Galois group of $N\colon M$ is M^*/N^*.

(Hint: Each proof should be a maximum of two sentences long.)