Math 306, Spring 2012
Homework 11 Solutions

(1) (5 pts/part)

(a) Show that the sequence of groups

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0,$$

where i and p are the canonical inclusion and projection, is exact.

(b) Prove the *Splitting Lemma*: For a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the following are equivalent:

(i) There exists a map $f' \colon B \to A$ such that $f' \circ f$ is the identity on A.

(ii) There exists a map $g' \colon C \to B$ such that $g \circ g'$ is the identity on C.

(iii) $B \cong A \oplus C$ (so this short exact sequence can be replaced by one from part (a)).

A sequence that satisfies these conditions is called *split exact*.

Solution:

(a) For exactness at A, it is clear that the kernel of i is only 0, so i is injective. Similarly it is clear that p is surjective. Kernel of p is A, which is precisely the image of i, so sequence is exact at $A \oplus C$.
Notice that i and p are the only maps that make this sequence exact.

(b) To show that (3) implies both (1) and (2), let f' be the natural projection of the direct sum onto A, and take as g' the natural injection of C into the direct sum.
To prove that (1) implies (3), first note that any member of B is in the set $\ker f' + \operatorname{im} f$. This follows since for all b in B, $b = (b - f(f'(b))) + f(f'(b))$; $f(f'(b))$ is obviously in $\operatorname{im} f$, and $b - f(f'(b))$ is in $\ker f'$ since

$$f'(b - f(f'(b))) = f'(b) - f'(f(f'(b))) = f'(b) - (f' \circ f)(f'(b)) = f'(b) - f(b) = 0.$$

Next, the intersection of $\operatorname{im} f$ and $\ker f'$ is 0, since if there exists an a in A such that $f(a) = b$, and $f'(b) = 0$, then $0 = f'(f(a)) = a$ and therefore $b = 0$. This proves that B is the direct sum of $\operatorname{im} f'$ and $\ker f$. So for all b in B, b can be uniquely written as $b = f(a) + k$ for some a in A and k in $\ker f'$.
By exactness, g is onto and so for any c in C there exists some $b = f(a) + k$ such that

$$c = g(b) = g(f(a) + k) = g(k).$$

Therefore for any c in C, there exists a k in $\ker f'$ such that $c = g(k)$. We also know that $g(\ker f') = C$. If $g(k) = 0$, then k is in $\operatorname{im} f$; since the intersection of $\operatorname{im} f$ and $\ker f' = 0$, $k = 0$. Therefore the restriction of the homomorphism g to $\ker f'$ is an isomorphism and so $\ker f'$ is isomorphic to C.
Finally, $\operatorname{im} f$ is isomorphic to A due to the exactness at A, and so B is isomorphic to the direct sum of A and C, which proves (3).
To show that (2) implies (3), use a similar argument. Any member of B is in the set $\ker g + \operatorname{im} g'$; since for all b in B, $b = (b - g'(g(b))) + g'(g(b))$, which is in $\ker g + \operatorname{im} g'$. The intersection of $\ker g$ and $\operatorname{im} g'$ is 0, since if $g(b) = 0$ and $g'(c) = b$, then $0 = g(g'((c)) = c$.
By exactness, $\operatorname{im} f = \ker g$, and since f is an injection, $\operatorname{im} f$ is isomorphic to A, so A is isomorphic to $\ker g$. Since $g \circ g'$ is a bijection, g' is an injection, and thus $\operatorname{im} g'$ is isomorphic to C. So B is again the direct sum of A and C.

(2) (5 pts) Suppose $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4$ is exact. Set

$$C_k = \ker(A_k \longrightarrow A_{k+1}) = \operatorname{im}(A_{k-1} \longrightarrow A_k) = \operatorname{coker}(A_{k-2} \longrightarrow A_{k-1})$$

and show that the sequences

$$0 \longrightarrow C_k \xrightarrow{f_k} A_k \xrightarrow{g_k} C_{k+1} \longrightarrow 0$$

are exact. This therefore gives an example of how an exact sequence can be broken into (and spliced from) short exact sequences (the picture of how this works was drawn in class).

Solution: For exactness at C_k, $\ker f_k$ is trivial since f_k is an inclusion.

For exactness at A_k, since f_k is an inclusion, $\operatorname{im} f_k = C_k$. Since g_k a projection from A_k to A_k/C_k, the elements of A_k that go to zero are precisely those in C_k. Thus $\ker g_k = C_k$.

For exactness at C_{k+1}, note that g_k surjects onto A_k/C_k (since it's the projection).

(3) (4 pts/part) For this problem, recall that by an extension we mean the entire exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, so that if there are two sequences with the same groups but different homomorphisms between them, we consider those extensions to be different.

(a) Show that any two extensions of C by A have the same number of elements, namely $|C| \cdot |A|$.

(b) Show that there are exactly two non-isomorphic extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$ and that there is exactly one extension of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.

(c) If p is a prime, show that there are exactly p non-isomorphic abelian extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$: the split extension and the extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where p is multiplication by p and i is the multiplication by i for $1 \leq i \leq p-1$.

Solution:

(a) Since B is an extension of C by A, we have $C = B/A$. By Lagrange's Theorem, $|C| = |B|/|A|$, which is what we want.

(b) In both cases, the extensions have to have order 6 by previous part. Besides the split extension, $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the only possibility is S_3. We will show that S_3 is indeed another extension of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$, while it cannot be an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$.

In the first case, we have a sequence $0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{f} S_3 \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. Elements of order 3 in $\mathbb{Z}/3\mathbb{Z}$ are 1 and 2 and they have to map to elements of order 3 in S_3, namely (123) and (132). Suppose $f(1) = (123)$ and $f(2) = (132)$ (the choice is thus not unique, but the two options give isomorphic extensions). Similarly, we need $g((123)) = g((132)) = 0$ since $\mathbb{Z}/2\mathbb{Z}$ has no elements of order 3. But then we have $\ker g = \operatorname{im} f$ and the sequence is exact.

For the second case, we have a sequence $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} S_3 \xrightarrow{g} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$. By the same reasoning as in the previous case, $f(1)$ must be an element of order two in S_3, say (12). On the other hand, $\ker g$ is a normal subgroup of S_3 so it can either be the identity or A_3. In neither of these case do we have that $\ker g = \operatorname{im} f$ (the sizes of the two sets are different).

(c) Any extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ has to have p^2 elements by part (a). The only groups of order p^2 are $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$. If the extension is $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then there are only the inclusion and the projection maps that will make the sequence exact.

If the extension is to be $\mathbb{Z}/p^2\mathbb{Z}$, then certainly the maps p and i as given in the problem make the sequence exact (image of p is $\{p, 2p, 3p, ..., (p-1)p\}$ and this is precisely the kernel of i for all $1 \leq i \leq p-1$).

These are the only possible extensions since i has to be what it is – in order to get an surjective map from $\mathbb{Z}/p^2\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}$, 1 cannot to go 0, and it can go to any other element, i.e. 1 can go to $i \cdot 1$ for any $1 \leq i \leq p-1$. This then forces p to be exactly what we have.

(4) (5 pts) Show that the sequence

$$\mathbb{Z}/8\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{d_1} \mathbb{Z}/4\mathbb{Z},$$

where each d_i is multiplication by 4, is a chain complex and compute its homology.

Solution: At $\mathbb{Z}/4\mathbb{Z}$, $\operatorname{im} d_3 = \{0\} \subset \ker d_2 = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z}$, and at $\mathbb{Z}/8\mathbb{Z}$, $\operatorname{im} d_2 = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z} \subset \ker d_1 = \mathbb{Z}/8\mathbb{Z}$. This is therefore a chain complex.

For homology, we get

$$\ker d_2/\operatorname{im} d_3 \cong (\mathbb{Z}/2\mathbb{Z})/\{0\} \cong \mathbb{Z}/2\mathbb{Z}, \qquad \ker d_1/\operatorname{im} d_2 \cong (\mathbb{Z}/8\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$$

(5) (5 pts) Recall the *Short Five Lemma* from lecture: Suppose the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

of abelian groups commutes and the rows are exact. Then
(a) α, γ injections $\Longrightarrow \beta$ is an injection;
(b) α, γ surjections $\Longrightarrow \beta$ is an surjection;
(c) α, γ isomorphism $\Longrightarrow \beta$ is an isomorphism;
Part (a) was proved in lecture. Prove part (b). (Note that part (c) follows from (a) and (b) immediately.)

Solution: Let b' be an element in B'. The image of b' in C' has a preimage in C, call it c. Since B maps onto C by exactness, let b be an element of B that maps to c.

By commutativity, the image of b in C' is the same regardless of how we go around the square. Thus $b' - \beta(b)$ must map to 0 in C'. By exactness, $b' - \beta(b)$ has some preimage a' in A'. Since α is surjective, let $a' = \alpha(a)$ for some a in A.

Finally consider $b+$(the image of a in B). Apply β, and b becomes $b' - (b' - \beta(b)) = \beta(b)$, while the image of a is $b' - \beta(b)$. The result is b', and β is surjective.

The remaining problems are optional and can be turned in for extra credit.

The solutions for these problems are straighforward and follow immediately from the definitions.

(6) (4 pts/part) Show that the following form a category:
(a) A group G (thought of as a category with one object and a morphism for each element);
(b) A poset P (where $X \leq Y$ means that there is a morphism from X to Y);

(7) (4 pts/part) Show that the following maps are functors.
(a) The map $F\colon \text{Sets} \to \text{Sets}$ which sends a set A to its power set $\mathcal{P}(A)$;
(b) The map $F\colon \text{Sets} \to \text{Grps}$ which sends a set A to the free group generated by the elements of A;

(8) (5 pts) For a fixed group H, show that the map $F_H\colon \text{Grps} \to \text{Grps}$ which sends a group G to $G \times H$ is a functor, and that each morphism (homomorphism) of groups $f\colon H \to K$ defines a natural transformation $F_H \to F_K$.

(9) (5 pts) If B and C are groups (regarded as categories with one object each) and $f, g\colon B \to C$ are functors (homomorphisms of groups), show that there is a natural transformation $f \to g$ if and only if f and g are conjugate, i.e. if and only if there is an element $c \in C$ with $g(b) = c(f(b))c^{-1}$ for all $b \in B$.

(10) (5 pts) Given two categories $\mathcal{C}$ and $\mathcal{D}$, one can form a new category $\mathcal{C} \times \mathcal{D}$, called the *product category*. An object of $\mathcal{C} \times \mathcal{D}$ is a pair (c, d) of objects $c \in Ob(\mathcal{C})$ and $d \in Ob(\mathcal{D})$ and a morphism $(c, d) \to (c', d')$ is a pair (f, g) of morphisms $f\colon c \to c'$ and $g\colon d \to d'$. The composition of such morphism is defined componentwise. Show that the product of categories includes the special cases of the product of groups and the product of sets.

(11) (7 pts) Two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are said to be *adjoint* if there is a function

$$\phi\colon \operatorname{Hom}_{\mathcal{D}}(F(-), -) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, G(-))$$

which to every pair of objects $c \in \mathcal{C}, d \in \mathcal{D}$ assigns a bijection

$$\phi_{c,d} \colon \operatorname{Hom}_{\mathcal{D}}(F(c), d) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(c, G(d)).$$

Show that the functor $\mathrm{Grps} \to \mathrm{Sets}$ which forgets the group structure and the functor $\mathrm{Sets} \to \mathrm{Grps}$ which to each set assigns the free group on that set are adjoint.