Math 306, Spring 2012
Homework 3 Solutions

(1) (2 pts/part) Each of the following statements is false. Disprove each of them by providing a counterexample or by appealing to the definitions.
(a) If K is a field, every polynomial in $K[t]$ has a root in K (recall that root is the same as a zero).
(b) Every polynomial which is irreducible in $\mathbb{Q}[t]$ is also irreducible in $\mathbb{R}[t]$.
(c) If K is a field and $f, g \in K[t]$ are coprime, then f and g have different degrees.
(d) If K is a field and $f \in K[t]$ has prime degree, then f is irreducible.
(e) If K is a field and $f \in K[t]$ has composite degree, then f is reducible.

Solution:
(a) The polynomial $p = x^2 + 1 \in \mathbb{Q}[x]$ has no root in $\mathbb{Q}$.
(b) The polynomial $p = x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{R}[x]$.
(c) The polynomials $p = x$ and $q = x + 1$ are coprime in $\mathbb{Q}[x]$ but have the same degree.
(d) The polynomial x^2 has degree 2 in $\mathbb{Q}[x]$, but is reducible.
(e) The polynomial $x^4 + x + 1$ has degree 4 in $\mathbb{Z}_2[x]$, but is irreducible.

(2) (3 pts/part) Determine whether each of the following is reducible or irreducible in the given polynomial ring. If reducible, write it as a product of its irreducible factors. State your reasons briefly.
(a) $t^3 - 5$ in $\mathbb{Z}_{11}[t]$
(b) $t^7 + 11t^3 - 33t + 22 \in \mathbb{Q}[t]$
(c) $t^4 + 1 \in \mathbb{R}[t]$

Solution:
(a) Since 3 is a root, we have $t^3 - 5 = (t - 3)(t^2 + 3t + 9)$. Since $t^2 + 3t + 9$ has no roots, it is irreducible.
(b) By Eisenstein's criterion with $p = 11$, this polynomial is irreducible.
(c) This polynomial is reducible, since $t^4 + 1 = (t^2 + \sqrt{2}t + 1)(t^2 - \sqrt{2}t + 1)$ in $\mathbb{R}[t]$.

(3) (4 pts/part) Prove that the following are all irreducible in the given polynomial ring.
(a) $x^6 + 539x^5 - 511x + 847$ in $\mathbb{Z}[x]$.
(b) $x^4 + x^3 + x^2 + 6x + 1$ in $\mathbb{Q}[x]$. (Hint: replace x with $x + 1$.)

Solution:
(a) Clearly the polynomial $x^6 + 539x^5 - 511x + 847$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein with $p = 7$. Therefore it is irreducible in $\mathbb{Z}[x]$.
(b) If $f = x^4 + x^3 + x^2 + 6x + 1$, then $f(x + 1) = x^4 + 5x^3 + 10x^2 + 15x + 10x$, which is irreducible in $\mathbb{Q}[x]$ by Eisenstein with $p = 5$. Hence f is also irreducible in $\mathbb{Q}[x]$.

(4) (4 pts/part) Let K be a field and let $f = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree $n \geq 1$.
(a) Prove that c is a root of f iff $f \in (x - c)$, the ideal generated by $(x - c)$. (Hint: use the Euclidean algorithm on f and $x - c$ and use the fact that $f \in (x - c)$ iff $f = (x - c)g$ for some $g \in K[x]$.)
(b) Use induction on $n = \deg f$ to prove that f has at most n distinct roots. Please mention unique factorization.

Solution:
(a) Suppose that $c \in K$ is a root of f. By the Euclidean algorithm, there are $q, r \in K[x]$ such that $f = (x - c)q + r$, where $\deg r < 1$, so r is a constant. Then $f(c) = (c - c)q(c) + r$, so $0 = r$. Then $f = (x - c)q$, so $f \in (x - c)$. Suppose that $f \in (x - c)$, so there is $q \in K[x]$ such that $f = (x - c)q$. Therefore $f(c) = (c - c)q(c) = 0$, so c is a root of f.
(b) If f has degree 1, then $f = a_0 + a_1x$ where a_1 is nonzero. Then $f(-a_0/a_1) = 0$, so $-a_0/a_1$ is a root. Clearly f has no other roots, so the claim is true for $\deg f = 1$. Suppose that any polynomial of degree

k has at most k distinct roots. Let f be a polynomial of degree $k+1$. If f has no roots in K, then we are done. If f has a root $c \in K$, then $f = (x-c)g$ for some $g \in K[x]$ by part (a). This factorization is unique and g has degree k. The roots of f are exactly the root of $x-c$ and the roots of g. Hence f has at most $k+1$ roots in K. If factorization were not unique, there might be a g', with its k roots, such that $f = (x-c)g'$, so f might potentially have $2k+1$ roots.