Math 306, Spring 2012
Homework 8 Solutions

(1) (4 pts/part) Determine the Galois group Gal(L/K) for each of these extensions $L\colon K$. Define the elements as precisely as possible. Here you should not assume that if $L\colon K$ is finite and normal, then the number of elements in Gal(L/K) is $[L\colon K]$.
  (a) $\mathbb{Q}(\sqrt{7})\colon \mathbb{Q}$
  (b) $\mathbb{Q}(\sqrt[5]{2})\colon \mathbb{Q}$
  (c) $\mathbb{Q}(e^{2\pi i/5})\colon \mathbb{Q}$
  (d) $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})\colon \mathbb{Q}$
  (e) $\mathbb{Z}_2(\zeta)\colon \mathbb{Z}_2$, where ζ is a root of $x^2+x+1 \in \mathbb{Z}_2[x]$

Solution:
  (a) We have Gal$(L/K) \cong \mathbb{Z}_2 = \{\sigma_1, \sigma_2\}$, where $\sigma_1\colon \sqrt{7} \mapsto \sqrt{7}$ and $\sigma_2\colon \sqrt{7} \mapsto -\sqrt{7}$.
  (b) We have Gal$(L/K) \cong \{e\}$ consisting of just the identity map.
  (c) We have Gal$(L/K) \cong \mathbb{Z}_4 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$. Let $\omega = e^{2\pi i/5}$. Then the automorphisms are determined by $\sigma_1\colon \omega \mapsto \omega$, $\sigma_2\colon \omega \mapsto \omega^2$, $\sigma_3\colon \omega \mapsto \omega^3$, $\sigma_4\colon \omega \mapsto \omega^4$.
  (d) We have Gal$(L/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, where the eight maps are given by the combinations $\sqrt{2} \mapsto \pm\sqrt{2}$, $\sqrt{3} \mapsto \pm\sqrt{3}$, $\sqrt{5} \mapsto \pm\sqrt{5}$.
  (e) We have Gal$(L/K) \cong \mathbb{Z}_2 = \{\sigma_1, \sigma_2\}$, where $\sigma_1\colon \zeta \mapsto \zeta$ and $\sigma_2\colon \zeta \mapsto \zeta + 1$.

(2) (3 pts/part) Let $\gamma = \sqrt{2+\sqrt{2}}$. The purpose of this problem is to compute the Galois group of $\mathbb{Q}(\gamma)\colon \mathbb{Q}$.
  (a) Compute the minimum polynomial $f \in \mathbb{Q}[x]$ of γ. Be sure to verify that f is indeed irreducible. Compute all the roots of f.
  (b) Let β be the other positive root of f. By showing that $\beta = \frac{\sqrt{2}}{\gamma}$, prove that $\mathbb{Q}(\gamma)$ is a splitting field for f over $\mathbb{Q}$.
  (c) By considering the order of the $\mathbb{Q}$-automorphism α satisfying $\alpha(\gamma) = \beta$ (we know there is one by Theorem 7), prove that Gal$(\mathbb{Q}(\gamma)/\mathbb{Q}) \cong \mathbb{Z}_4$.

Solution:
  (a) We know that $\gamma^2 = 2+\sqrt{2}$, so $(\gamma^2-2)^2 = 2$, or $\gamma^2 - 4\gamma + 2 = 0$. Therefore $f = x^2 - 4x + 2$ has γ as a root. Since f is irreducible by Eisenstein's criterion, it is the minimum polynomial of γ. By the quadratic equation, one can easily see that the four roots of f are given by $\pm\sqrt{2\pm\sqrt{2}}$.
  (b) Let $\beta = \sqrt{2-\sqrt{2}}$. Note that $\gamma\beta = \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} = \sqrt{2}$, so it follows that $\beta = \frac{\sqrt{2}}{\gamma}$. Clearly $\sqrt{2} \in \mathbb{Q}(\gamma)$, so $\beta \in \mathbb{Q}(\gamma)$ as well. Since the roots of f are $\pm\gamma$ and $\pm\beta$, it follows that $\mathbb{Q}(\gamma)$ is the splitting field of f.
  (c) There are four elements of the Galois group $G = $ Gal$(\mathbb{Q}(\gamma)\colon \mathbb{Q})$. Consider $\alpha \in G$ determined by $\gamma \mapsto \beta$. Then $\alpha(\sqrt{2}) = -\sqrt{2}$. Therefore $\alpha^2(\gamma) = \alpha(\beta) = \alpha(\sqrt{2}/\gamma) = -\sqrt{2}/\beta = -\gamma$. Therefore the order of α is 4, so $G \cong \mathbb{Z}_4$.

(3) Given $f \in K[x]$, we say the that Galois group of f is the Galois group of the extension $L : K$ where L is the splitting field of f over K. Consider $f = (x^2-2)(x^2-3)(x^2-5) \in \mathbb{Q}[x]$.
  (a) (3 pts) Determine the Galois group G of f, listing all its elements using the $\mapsto$ notation.
  (b) (4 pts) For each subgroup H of G, compute $H^\dagger$.

Solution:

(a) The roots of f are given by $\pm\sqrt{2}$, $\pm\sqrt{3}$ and $\pm\sqrt{5}$. Every automorphism in the Galois group of f is determined by its behavior on $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. The eight automorphisms of f are given by

$$\begin{array}{lllll}
\sigma_1: & \sqrt{2}\mapsto\sqrt{2}, & \sqrt{3}\mapsto\sqrt{3}, & \sqrt{5}\mapsto\sqrt{5}, & (0,0,0); \\
\sigma_2: & \sqrt{2}\mapsto-\sqrt{2}, & \sqrt{3}\mapsto\sqrt{3}, & \sqrt{5}\mapsto\sqrt{5}, & (1,0,0); \\
\sigma_3: & \sqrt{2}\mapsto\sqrt{2}, & \sqrt{3}\mapsto-\sqrt{3}, & \sqrt{5}\mapsto\sqrt{5}, & (0,1,0); \\
\sigma_4: & \sqrt{2}\mapsto\sqrt{2}, & \sqrt{3}\mapsto\sqrt{3}, & \sqrt{5}\mapsto-\sqrt{5}, & (0,0,1); \\
\sigma_5: & \sqrt{2}\mapsto-\sqrt{2}, & \sqrt{3}\mapsto-\sqrt{3}, & \sqrt{5}\mapsto\sqrt{5}, & (1,1,0); \\
\sigma_6: & \sqrt{2}\mapsto\sqrt{2}, & \sqrt{3}\mapsto-\sqrt{3}, & \sqrt{5}\mapsto-\sqrt{5}, & (0,1,1); \\
\sigma_7: & \sqrt{2}\mapsto-\sqrt{2}, & \sqrt{3}\mapsto\sqrt{3}, & \sqrt{5}\mapsto-\sqrt{5}, & (1,0,1); \\
\sigma_8: & \sqrt{2}\mapsto-\sqrt{2}, & \sqrt{3}\mapsto-\sqrt{3}, & \sqrt{5}\mapsto-\sqrt{5}, & (1,1,1).
\end{array}$$

The last column gives the corresponding element of $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

(b) There are a total of 16 subgroups of the Galois group.

H	$H^\dagger$	H	$H^\dagger$
$\langle\sigma_1\rangle$	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$	$\langle\sigma_2,\sigma_3\rangle$	$\mathbb{Q}(\sqrt{5})$
$\langle\sigma_2\rangle$	$\mathbb{Q}(\sqrt{3},\sqrt{5})$	$\langle\sigma_2,\sigma_4\rangle$	$\mathbb{Q}(\sqrt{3})$
$\langle\sigma_3\rangle$	$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$\langle\sigma_3,\sigma_4\rangle$	$\mathbb{Q}(\sqrt{2})$
$\langle\sigma_4\rangle$	$\mathbb{Q}(\sqrt{2},\sqrt{3})$	$\langle\sigma_2,\sigma_6\rangle$	$\mathbb{Q}(\sqrt{15})$
$\langle\sigma_5\rangle$	$\mathbb{Q}(\sqrt{5},\sqrt{6})$	$\langle\sigma_3,\sigma_7\rangle$	$\mathbb{Q}(\sqrt{10})$
$\langle\sigma_6\rangle$	$\mathbb{Q}(\sqrt{2},\sqrt{15})$	$\langle\sigma_4,\sigma_5\rangle$	$\mathbb{Q}(\sqrt{6})$
$\langle\sigma_7\rangle$	$\mathbb{Q}(\sqrt{3},\sqrt{10})$	$\langle\sigma_5,\sigma_6\rangle$	$\mathbb{Q}(\sqrt{30})$
$\langle\sigma_8\rangle$	$\mathbb{Q}(\sqrt{6},\sqrt{10},\sqrt{15})$	$\langle\sigma_1,\sigma_2,\sigma_3\rangle$	$\mathbb{Q}$

(4) (4 pts/part) Find the Galois group of x^3-5 over the following fields. List all the subgroups H and the fixed field $H^\dagger$.

(a) $\mathbb{Q}$
(b) $\mathbb{Z}_3$
(c) $\mathbb{Z}_7$

Solution:

(a) The Galois group is S_3, each automorphism determined by its behavior on $\sqrt[3]{5}$ and $\omega = e^{2\pi i/3}$. The six elements are as follows:

$$\begin{array}{lll}
\sigma_1: & \sqrt[3]{5}\mapsto\sqrt[3]{5}, & \omega\mapsto\omega; \\
\sigma_2: & \sqrt[3]{5}\mapsto\sqrt[3]{5}\,\omega, & \omega\mapsto\omega; \\
\sigma_3: & \sqrt[3]{5}\mapsto\sqrt[3]{5}\,\omega^2, & \omega\mapsto\omega; \\
\sigma_4: & \sqrt[3]{5}\mapsto\sqrt[3]{5}, & \omega\mapsto\omega^2; \\
\sigma_5: & \sqrt[3]{5}\mapsto\sqrt[3]{5}\,\omega, & \omega\mapsto\omega^2; \\
\sigma_5: & \sqrt[3]{5}\mapsto\sqrt[3]{5}\,\omega^2, & \omega\mapsto\omega^2.
\end{array}$$

There are six subgroups of S_3.

H	$H^\dagger$
$\langle\sigma_1\rangle$	$\mathbb{Q}(\sqrt[3]{5},\omega)$
$\langle\sigma_2\rangle$	$\mathbb{Q}(\omega)$
$\langle\sigma_4\rangle$	$\mathbb{Q}(\sqrt[3]{5})$
$\langle\sigma_5\rangle$	$\mathbb{Q}(\sqrt[3]{5}\omega^2)$
$\langle\sigma_6\rangle$	$\mathbb{Q}(\sqrt[3]{5}\omega)$
$\langle\sigma_2,\sigma_4\rangle$	$\mathbb{Q}$

(b) Since $f=(x+1)^3$, it follows that the Galois group is $\langle e\rangle$ and $\langle e\rangle^\dagger=\mathbb{Z}_3$.

(c) Note first that f is irreducible over $\mathbb{Z}_7$. Let γ be a root. Then $f = (x-\gamma)(x-2\gamma)(x-4\gamma)$. Hence the Galois group is $\mathbb{Z}_3$. Hence $\langle e \rangle^\dagger = \mathbb{Z}_7(\gamma)$ and $\mathbb{Z}_3^\dagger = \mathbb{Z}_7$.