Math 307, Fall 2010

Homework 2, due Friday, September 24

(1) What are the limit points of $\mathbb{Q} \subset \mathbb{R}$? (This should be a one-line answer; it's ok to use something you know from analysis about how the rationals distribute among all the reals.)

(2) (p.31, 1(a)(b)(c)) Let A and B be subsets of a topological space. Show that:
  (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
  (b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
  (c) $\overline{\overline{A}} = \overline{A}$

(3) (p.31, 2) Show that there exists a family of closed subsets of the real line whose union is not closed.

(4) (p.31, 3(c)) Let $B = \{(x, \sin(1/x) : x > 0\}$, $C = \{(0, y) : y \in [-1, 1]\}$ and $A = \mathbb{R}^2 - B$. Show that:
  (a) $A^\circ = A - C$ (A° is the interior of A);
  (b) $\overline{A} = \mathbb{R}^2$ ($\overline{A}$ is the interior of A);
  (c) $\partial A = (A^\circ)^c = B \cup C$ (∂A is the boundary of A).

(5) (p.31, 9) Let $Y \subset X$ be a subspace. Suppose that A is open (closed) in Y, and Y is open (closed) in X. Show that A is open (closed) in X.

(6) Show that A° is the largest open set contained in A, i.e. show that any open set contained in A is contained in A°. (Hint: Mimic the proof that $\overline{A}$ is the smallest closed set contaning A.)

(7) Show the following.
  (a) $A^\circ \cap \partial A = \emptyset$;
  (b) $A \cup \partial A = \overline{A}$.
  (c) $\partial A = \emptyset$ if and only if A is open and closed.

(8) Determine the closure of the set $K = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ in each of the following topologies on $\mathbb{R}$.
  (a) Standard topology;
  (b) The *finite complement topology* given by all subsets U of $\mathbb{R}$ such that $\mathbb{R} - U$ is either finite or all of $\mathbb{R}$;
  (c) The *upper limit topology* with all intervals $(a, b]$ as basis.

(9) (p.35, 13)
  (a) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a map, the set of points fixed by f is closed in $\mathbb{R}$.
  (b) Show that if g is a continuous real-valued function on X, the set $\{x \mid g(x) = 0\}$ is closed.

(10) (p.35, 15) Let $f : \mathbb{R} \to \mathbb{R}$ be a map and define its graph $\Gamma_f : \mathbb{R} \to \mathbb{R}^2$ by $\Gamma_f(x) = (x, f(x))$. Show that Γ_f is continuous and its image (with topology induced from $\mathbb{R}^2$) is homeomorphic to $\mathbb{R}$. (Hint: You can take open rectangles (rather than discs) as basis for the standard topology on $\mathbb{R}^2$.)

(11) (p.36, 20(a)(b)) A map is *open* (*closed*) if it sends open (closed) sets to open (closed) sets.
  (a) Show that $f(x) = e^{ix}$ from the line to the circle is open and closed.
  (b) Show that $f(x, y) = (x, |y|)$ is closed but not open.