Math 307, Fall 2008
Homework 3, due Friday, October 1

(1) (p.22, 10) Find a homeomorphism from the real line to the interval $(0, 1)$ and show any two open intervals are homeomorphic.

(2) (p.22, 11) Each of the following pairs of spaces can be continuously deformed into one another. Draw a few pictures describing the deformation in each case. Pictures of the objects are given in the book:
  (a) The cylinder with a disk removed and the disk with two holes.
  (b) The torus with a disk removed and two cylinders glued together over a square patch.
  (c) The double torus and the double torus with arms linked.

(3) (p.22, 13) Let x and y be points on the sphere. Describe a homeomorphism of the sphere with itself which takes x to y. Repeat with the sphere replaced by the plane and by the torus. (You do not need to write down formulas for the maps; just describe the homeomorphism in words and draw some pictures if that helps.)

(4) (p.41, 27) If d is a metric on X, A is a subset of X, and $x \in X$, we define the *distance from x to A* to be

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Show that $d(x, A) = 0$ iff x is a point of $\overline{A}$.

(5) (p.41, 29) Given any set X, show that the function

$$d(x, y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y \end{cases}$$

defines a metric and that the topology given by this metric is the discrete topology.

(6) Show that if $\{B_i\}_{i=1}^{\infty}$ is a sequence of boxes such that $B_i \supset B_{i+1}$ for all i and such that

$$\lim_{i \to \infty} \text{diameter}(B_i) = 0,$$

then $\bigcap_{i=1}^{\infty} B_i$ is a single point (we needed this in the proof of the Heine-Borel Theorem). (Hint: Use completeness of $\mathbb{R}^n$, i.e. the fact that $\mathbb{R}^n$ contains limits of all its convergent sequences.)

(7) (p.49, 5) Which of the following are compact? Give brief explanations.
  (a) The space of rational numbers.
  (b) S^n with a finite number of points removed.
  (c) The torus with an open disc removed.
  (d) The Klein bottle (see pages 9–10 for the definition of this space).
  (e) The Möbius strip with its boundary circle removed.

(8) (p.50, 11) Give an example of a topological space with a subset which is compact but whose closure is not compact.