Math 307, Fall 2008

Homework 4, due Friday, October 8

(1) (p.50, 14) Let f be a one-to-one map from a compact space X to a Hausdorff space Y. Show that f is an *embedding* of X into Y, i.e. $f : X \to f(X)$ is a homeomorphism.

(2) (p.50, 15): A space is *locally compact* if each of its points has a compact neighborhood. Show the following:
  (a) Any compact space is locally compact.
  (b) $\mathbb{R}^n$ is locally compact.
  (c) Any discrete space (a space with discrete topology) is locally compact.
  (d) Any closed subset of a locally compact space is locally compact.
  (e) $\mathbb{Q} \subset \mathbb{R}$ is not locally compact.
  (f) Local compactness is preserved by homeomorphisms.

(3) (p.55, 20) Show that the following set equalities hold.
  (a) $\overline{A \times B} = \overline{A} \times \overline{B}$;
  (b) $(A \times B)^\circ = A^\circ \times B^\circ$;
  (c) $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.

(4) (p.55, 25) Given a space X, consider the *diagonal map* $\Delta : X \to X \times X$ defined by $\Delta(x) = (x, x)$
  (a) Show that Δ is continuous.
  (b) Show that a topological space X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

(5) (p.55, 26) We showed in class that the projection maps are open. Are they also closed?

(6) Let X be a metric space with metric d. If $f\colon X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an *isometry* of X. Show that it f is an isometry and X is compact, then f is bijective and hence a homeomorphism. (Hint: If $a \notin f(X)$, choose ϵ such that the ϵ-ball around a is disjoint from $f(X)$. Set $x_1 = a$ and define $x_{n+1} = f(x_n)$. Show that $d(x_n, x_m) \geq \epsilon$ for $n \neq m$. The reason we are creating a sequence whose points are always "far apart" is so that we can use the Bolzano-Weierstrass property and contradict the assumption that X is compact.)