Math 307, Fall 2010
Homework 6, due Friday, October 29

(1) (p.78, 16) Show that $O(n)$ and $SO(n) \times \mathbb{Z}_2$ are homeomorphic. (It turns out they are also isomorphic as groups for n odd, but not for n even as they have centers of different sizes; center of $O(n)$ has two elements while center of $SO(n) \times \mathbb{Z}_2$ has four elements.)

(2) (p.78, 21) Show that every nontrivial discrete subgroup of $\mathbb{R}$ with addition is infinite cyclic, i.e. it is isomorphic to $\mathbb{Z}$. (Hint: Feel free to use Theorem 4.11 from the book.)

(3) (p.85, 26) Find an action of $\mathbb{Z}$ on $\mathbb{R} \times [0, 1]$ which has the Möbius strip as the orbit space.

(4) (p.85, 27) Find an action of $\mathbb{Z}_2$ on T^2 in the (x, y)-plane with the cylinder as the orbit space. (Here T^2 is thought of as sitting in $\mathbb{R}^3$ in such a way that z-axis points through the hole.)

(5) Let G be a group acting on a space X. The stabilizer of any point $x \in X$ consists of those group elements g in G for which $g(x) = x$. Show that the stabilizer of any point is a closed subgroup of G when X is Hausdorff, and points in the same orbit have conjugate stabilizers for any x.

(6) (a) (Munkres p.199, 1) Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.
(b) (Munkres p.199, 2) Show that if X is normal, every pair of disjoint closed subsets of X have neighborhoods whose closures are disjoint.

(7) (Munkres p.205, 3) Show that every locally compact Hausdorff space is regular.

(8) (Munkres p.212, 3) Give a direct proof of the Urysohn Lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

(9) We say that a subspace Y of X is a *retract of* X if there exists a map $r \colon X \to Y$ such that $r(y) = y$ for each $y \in Y$.
(a) (Munkres p.223, 4(a)) Show that if X is Hausdorff and Y is a retract of X, then Y is closed in X.
(b) (Munkres p.224, 7(b)) Show that the "knotted x-axis" (the right picture in Figure 35.2) is a retract of $\mathbb{R}^3$. (Hint: First observe that the knot given in the picture is actually homeomorphic to the real line. You don't have to write down an explicit homeomorphism, but you should try to describe one. Then use Tietze Extension Theorem.)