Math 307, Fall 2010

Homework 7, due Friday, November 5

(1) (p.91, 1) Let S^1 denote the unit circle in $\mathbb{R}^2$ as usual. Suppose $f : S^1 \to S^1$ is a map which is not homotopic to the identity map. Show that $f(x) = -x$ for some point x of S^1.

(2) A space X is *contractible* if the identity map $Id_X \colon X \to X$ given by $Id_X(x) = x$ for all $x \in X$ is i.e. f is *nullhomotopic*, i.e. it is homotopic to a map which takes all of X to a single point of X.
  (a) Show that I and $\mathbb{R}^n$ are contractible.
  (b) Show that a contractible space is path connected.
  (c) Show that for any X and a contractible Y, $[X, Y]$ has only one element. Note that this in particular means that $\pi_1(Y)$ is trivial for contractible Y.
  (d) Show that for X contractible and Y path connected, $[X, Y]$ has only one element.

(3) (p.91, 5)
  (a) Let $f : X \to S^n$ be a map which is not onto. Show that f is nullhomotopic (see previous problem for what this means).
  (b) Use part (a) to show that $\pi_1(S^n)$, $n > 1$, is trivial.

(4) (p.91, 6) Suppose X, Y are any spaces and denote by CY the cone on Y as usual. Show that $[X, CY]$ has only one element (i.e. all maps from X to the cone on Y are homotopic).

(5) Recall that our proof of associativity of the product of loops in a space X based at p used a reparametrization of the interval $f \colon I \to I$ given by

$$f(t) = \begin{cases} 2t, & 0 \le t \le 1/4; \\ t + 1/4, & 1/4 \le t \le 1/2 \\ (t+1)/2, & 1/2 \le t \le 1. \end{cases}$$

To complete that proof, do the following:
  (a) Explain how, for any three loops α, β, and γ in X, $\alpha(\beta\gamma) \circ f = (\alpha\beta)\gamma$.
  (b) Show that reparametrizing a loop preserves its homotopy class, i.e. show that for any loop $l \colon I \to X$ and a reparametrization $f \colon I \to I$, $l \circ f \simeq l$. (Using this and part (a), we thus deduce that $(\alpha\beta)\gamma = \alpha(\beta\gamma) \circ f \simeq \alpha(\beta\gamma)$ which completes the proof of associativity.)

(6) (p.95, 12) Show that any indiscrete space (a space whose topology consists only of the empty set and the entire space) has trivial fundamental group.