Math 307, Fall 2010

Homework 8, due Friday, November 12

(1) (p.102, 1) Use Theorem 5.13 to show that the Möbius strip and the cylinder both have fundamental group $\mathbb{Z}$.

(2) Let p and q be relatively prime integers (not necessarily prime).

  (a) Show that S^3 can be thought of as the unit sphere in the complex space of dimension 2 by setting

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid z_0\overline{z_0} + z_1\overline{z_1} = 1\}.$$

  Here $\overline{z}$ means the conjugate of the complex number $z = x + iy$, i.e. $\overline{z} = x - iy$.

  (b) Let g be the generator of the cyclic group $\mathbb{Z}_p$ and define an action of $\mathbb{Z}_p$ on S^3 by

$$g(z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi qi/p} z_1).$$

  Show that this is indeed an action. The resulting orbit space $S^3/\mathbb{Z}_p$ is called a *lens space* and denoted by $L(p, q)$.

  (c) Deduce that $\pi_1(L(p, q)) = \mathbb{Z}_p$.

  (d) Show that, for any finite abelian group G, there exists a space X such that $\pi_1(X) = G$. (Hint: Use the Structure Theorem for Finite Abelian Groups.)

(3) Use fundamental groups to show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$. You may assume that fundamental group is a homeomorphism invariant (we'll show this in class). (Hint: Argue that $S^{n-1} \times (0, 1)$ is homeomorphic to $\mathbb{R}^n \setminus \{\text{point}\}$.)

(4) Recall from class that for a path-connected space X, there is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ for any two points x_0, x_1 in X. The isomorphism is given by the map $\hat{\alpha}$ which sends $\langle f \rangle$ to $\langle \alpha^{-1} f \alpha \rangle$, where α is a path between x_0 and x_1. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 and x_1, we have $\hat{\alpha} = \hat{\beta}$.