NAME: SOLUTIONS

Instructions: Check that your test has 11 pages, including this one and the blank one on the bottom. There are 9 problems on the exam. *Write neatly:* solutions deemed illegible will not be graded, so no credit will be given. You must show all work, justify all nonobvious parts of your work, and reference theorems or other facts you know from class or textbook in order to receive credit. Use full English sentences. This exam is closed book, closed notes. Calculators are not allowed.

PLEDGE: On my honor as a student, I have neither given nor received aid on this exam.

SIGNATURE: ______________________

1. (12 points) ____________
2. (12 points) ____________
3. (5 points) ____________
4. (5 points) ____________
5. (5 points) ____________
6. (5 points) ____________
7. (5 points) ____________
8. (5 points) ____________
9. (6 points) ____________

Total (out of 60): ____________

Part I: Things you've seen before

1. (3 pts each) Write down precise definitions of the following:

   (a) Countable set:

   See Definition 1.4.10

   (b) Cauchy sequence:

   See Definition 2.6.1

   (c) Continuous function:

   See Definition 4,3.1

   (d) Riemann-integrable function:

   See Definition 7.2.7

2. (3 pts each) State the following theorems:

(a) Bolzano-Weierstrass Theorem:

See Theorem 2.5.5.

(b) Mean Value Theorem:

See Theorem 5.3.2.

(c) L'Hospital's Rule, 0/0 case:

See Theorem 5.3.6.

(d) Cauchy Criterion for Uniform Convergence of a Series of Functions:

See Theorem 6.4.4.

3. (5 pts) Prove the Monotone Convergence Theorem.

See Theorem 2.4.2.

4. (5 pts) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable? Justify your answer.

See solution for Problem 4 on the midterm exam.

(Even though this problem was thrown out of the midterm, we went over it in class.)

5. (5 pts) Prove the Weierstrass M-Test (you may assume the Cauchy Criterion for Uniform Convergence of a Series).

This was exercise 6.4.2

Suppose (f_n) is a sequence of functions defined on a set A and suppose

$$\circledast \qquad |f_n(x)| \leq M_n \quad \forall x \in A.$$

Also suppose $\sum M_n$ converges.

Then given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$$|M_{m+1} + M_{m+2} + \cdots + M_n| < \varepsilon \quad \forall m > n > N \qquad (\text{Thm } 2.7.2)$$

But then

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)|$$
$$\leq |M_{m+1} + M_{m+2} + \cdots + M_n| < \varepsilon \quad \forall m > n > N$$

and so $\sum f_n(x)$ converges uniformly by the Cauchy criterion (6.6.4.)

6. (5 pts) Suppose $f\colon [a,b] \to \mathbf{R}$ is increasing on $[a,b]$. Show f is Riemann-integrable on $[a,b]$.

This was exercise 7.2.6

Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

$$\sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_k)$$

and $$\inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_{k-1})$$

because f is increasing. So

$$U(f,P) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1})$$

$$L(f,P) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

Now let P be the partition of $[a,b]$ into n equal intervals, so

$$(x_k - x_{k-1}) = \frac{b-a}{n}.$$

Then

$$U(f,P_n) - L(f,P_n) = \frac{b-a}{n} \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))$$

$$= \frac{b-a}{n}(f(x_n) - f(x_0))$$

$$= \frac{b-a}{n}(f(b) - f(a))$$

Choosing n large, this can be as small as we want, and so f is integrable by Thm 7.2.8.

Part II: Things you haven't seen before

7. (5 pts) A function $f\colon A \to \mathbf{R}$ is called *Lipshitz* if there exists a number M such that

$$|f(x) - f(y)| \le M|x - y|$$

for all x, y in A. Now suppose $A = [a, b]$ and f is differentiable with the derivative f' which is continuous on A. Show f is Lipshitz.

f' continuous, so by Extreme Value Theorem, it attains max/min on $[a,b]$ and is thus bounded, so

$\exists M > 0$ s.t. $|f'(x)| \le M \quad \forall x \in [a,b]$.

By Mean Value Theorem, $\forall x, y \in [a,b] \ \exists c$ s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

$$\Rightarrow \quad \left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \le M \quad \forall x, y, c$$

$$\Rightarrow \quad |f(x) - f(y)| \le M|x - y|$$

8. (5 pts) Given $f : \mathbf{R} \to \mathbf{R}$, define a sequence of functions (f_n) by $f_n(x) = f(x + \frac{1}{n})$. Prove that if f is uniformly continuous, then the sequence (f_n) converges uniformly to f.

f uniformly continuous

$\Rightarrow$ Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y.$

Let $N = \frac{1}{\delta}$. Then

$$\underbrace{|x - (x + \tfrac{1}{n})|}_{= \frac{1}{n}} < \delta$$

$\Rightarrow |f(x) - f(x + \frac{1}{n})| < \varepsilon \quad \forall n > N.$

So $|f(x) - f_n(x)| < \varepsilon \quad \forall n > N, \ \forall x,$

which is exactly what we want.

9. (5 pts) Let

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Riemann-integrable on $[0,1]$ and compute $\int_0^1 f$.

From the definition, we always have

$L(f) \le U(f)$ for any f.

Since our f is always nonnegative, we further have

$0 \le L(f) \le U(f)$.

The claim is that $\underline{0 = L(f) = U(f)}$

$\Rightarrow$ f is integrable <u>and</u> $\int_a^b f = 0$.

To show this, it suffices to show $U(f)$ can be made arbitrarily small:

Given $\varepsilon > 0$, choose $n > \frac{1}{\varepsilon}$ and change the values of f for $x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ from 1 to 0.

(OK by exercise 7.3.4 (b)).

Then letting P_n be the partition into n equal parts gives

$U(f, P_n) < \varepsilon$.