MATH 331 MIDTERM EXAM SPRING 2006

NAME: SOLUTIONS

Instructions: Check that your test has 11 pages, including this one and the blank one on the bottom. There are 8 problems on the exam. *Write neatly:* solutions deemed illegible will not be graded, so no credit will be given. You must show all work, justify all nonobvious parts of your work, and reference theorems or other facts you know from class or textbook in order to receive credit. Use full English sentences. This exam is closed book, closed notes. Calculators are not allowed.

PLEDGE: On my honor as a student, I have neither given nor received aid on this exam.

SIGNATURE: ______________________

1. (12 points) ________
2. (5 points) ________
3. (5 points) ________
4. ~~(5 points)~~ ________
5. (12 points) ________
6. (10 points) ________
7. (10 points) ________
8. (11 points) ________

Total (out of ~~70~~): ________
65

Part I: Things you've seen before

Note: All these theorems were also stated in class.

1. (3 pts each) State the following theorems:

   (a) Nested Interval Property:

   See Thm 1.4.1.

   (b) Bolzano-Weierstrass Theorem:

   See Thm 2.5.5.

   (c) Heine-Borel Theorem:

   See Thm 3.3.4.

   (d) Intermediate Value Theorem:

   See Thm 4.5.1.

2. (5 pts) Prove the Monotone Convergence Theorem.

See Thm 2.4.2.

This was also done in class.

3. (5 pts) Prove that if a set $K \subset \mathbf{R}$ is closed and bounded, then it is compact.

This is one direction of Heine-Borel Thm, which you did in Exercise 3.3.2 for homework.

Let (x_n) be a sequence in K.
Since K is bounded, (x_n) has a convergent subsequence (x_{n_k}).
Since K is closed, $\lim x_{n_k}$ is in K.
Thus K is compact.

This problem was thrown out since it is something you had not actually seen before.

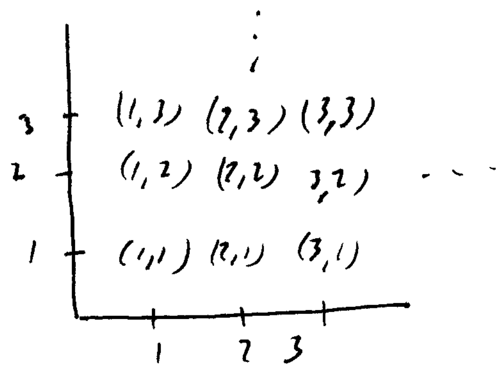
4. (5 pts) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable? Justify your answer.

This was exercise 1.5.9 (a) on homework.

To have a function $f : \{0,1\} \longrightarrow \mathbb{N}$ is the same as specifying a pair of natural numbers $(f(0), f(1))$. Thus the set of all such functions is in one-to-one correspondence with the set

$$A = \{ (a,b) : a, b \in \mathbb{N} \}$$

This set is <u>countable</u>. For example, arranging the pairs (a,b) into an array



can be used for the correspondence between A and $\mathbb{N}$.

5. (3 pts each) Give an example of each of the following or argue that such a request is impossible:

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

This was Exercise 2.5.3 (e)

Impossible: If a subsequence (x_{n_k}) of a sequence is bounded, then it contains a convergent subsequence $(x_{n_{k_\ell}})$, which is itself a subsequence of (x_n).

(b) A Cauchy sequence with a divergent subsequence.

This was Exercise 2.6.1 (c)

Impossible: Cauchy $\iff$ convergent and all subsequences of a convergent sequence converge.

(c) A series $\sum a_n$ which converges absolutely but $\sum a_n^2$ does not.

This was Exercise 2.7.5 (a)

Impossible: Σ converges absolutely

$\Rightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \varepsilon \quad \forall m, n > N$.

But then, given $\varepsilon^2 > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|a_{m+1}^2| + |a_{m+2}^2| + \cdots + |a_n^2| = |a_{m+1}^2 + a_{m+2}^2 + \cdots + a_n^2|$$

$$\leq (|a_{m+1}| + |a_{m+2}| + \cdots + |a_n|)^2 < \varepsilon^2 \quad \forall n > m > N.$$

(d) A function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ which is discontinuous for all $0 < x < 1$ and continuous otherwise.

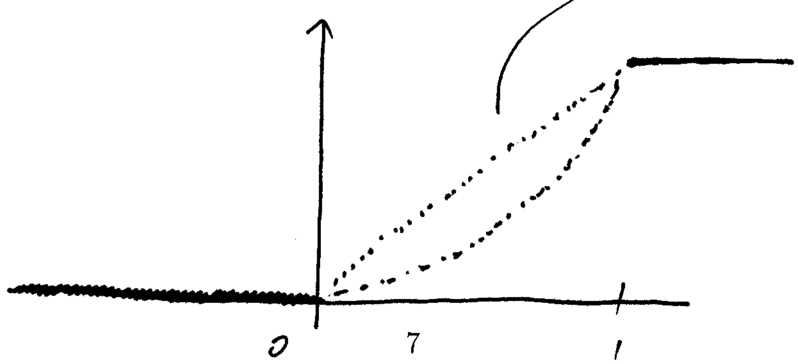
This was Exercise 4.3.11 (b)

Let

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \in (0,1) \cap \mathbb{Q} \\ x^2, & x \in (0,1) \cap \mathbb{I} \\ 1, & x \geq 1 \end{cases}$$

like Dirichlet's function.

picture:



Part II: Things you haven't seen before

6. (a) (5 pts) Show that if $\sum x_n$ converges absolutely and (y_n) is a bounded sequence, then the sum $\sum x_n y_n$ converges.

(y_n) bounded $\Rightarrow \exists M \in \mathbb{N}$ s.t. $|y_n| \leq M \ \forall n \in \mathbb{N}$.

$\Rightarrow |x_n y_n| \leq M|x_n| \quad \forall n \in \mathbb{N}$

Then $\sum x_n$ absolutely convergent

$\Rightarrow \sum M x_n$ absolutely convergent (Thm 2.3.3. (i))

$\Rightarrow \sum x_n y_n$ absolutely convergent by the Comparison Thm.

(b) (5 pts) Find a counterexample that demonstrates that part (a) does not always hold if the convergence of $\sum x_n$ is conditional.

Let $(x_n) = \left((-1)^n \frac{1}{n}\right)$, $(y_n) = \left((-1)^n\right)$.

Then $\sum x_n$ converges (but not absolutely)

and $((-1)^n)$ is bounded, <u>but</u>

$\sum x_n y_n = \sum \frac{1}{n}$ diverges.

7. (a) (5 pts) Using the definition of continuity, show that if $f\colon \mathbf{R} \to \mathbf{R}$ is continuous at c and $f(c) > 0$, then there exists a $\delta > 0$ such that $f(x) > 0$ for all x between $c - \delta$ and $c + \delta$.

use definition of continuity with $\varepsilon = f(c)$.
Then we have that $\exists\ \delta > 0$ s.t.

$$|x-c| < \delta \implies |f(x) - f(c)| < f(c)$$
$$\Updownarrow$$
$$0 < f(x) < 2f(c)$$

So if $f(c) > 0$, then $f(x) > 0$ as well.

(b) (5 pts) Use the result from part (a) to show that if $f\colon \mathbf{R} \to \mathbf{R}$ is continuous, the set $\{x\colon f(x) > 0\}$ is open.

Let $A = \{x : f(x) > 0\}$. If $c \in A$, then (a) says that $\exists\ \delta > 0$ s.t.

$$V_\delta(c) = (c - \delta, c + \delta) \subseteq A$$ (see definition of continuity)

But this means A is open by Def 3.2.1.

8. (a) (4 pts) Give an example of a continuous function $f\colon (0,\infty) \to \mathbf{R}$ and a bounded subset $B \subset (0,\infty)$ such that $f(B)$ is not a bounded set.

Let $f(x) = \frac{1}{x}$, $B = (0,1)$.

f is continuous on $(0,\infty)$, B is bounded, but

$f(B) = (1,\infty)$ is <u>not</u> <u>bounded</u>.

(b) (4 pts) Give an example of a continuous function $f\colon (0,\infty) \to \mathbf{R}$ and a closed subset $C \subset (0,\infty)$ such that $f(C)$ is not a closed set.

Let $f(x) = \frac{1}{x}$, $C = [1,\infty)$.

f is continuous, C closed, but

$f(C) = (0,1]$ is <u>not</u> <u>closed</u>.

(c) (3 pts) If you did parts (a) and (b) correctly, your B is not closed and C is not bounded. Why was this to be expected?

If B or C were closed <u>and</u> bounded, they would be compact (Heine-Borel) and thus their image would also be compact (Thm 4.4.2) and thus closed and bounded.