Math 349 Algebraic Geometry, Spring 2009
Homework 9, due Friday, April 24

(1) Chapter 8, §4, problem 6

(2) Chapter 8, §4, problem 9

(3) Chapter 8, §4, problem 13

(4) Given an ideal $I \subset k[x_1, ..., x_n]$, show that the relation of congruence mod I is an equivalence relation.

(5) Show that $k[x, y]/\mathbb{I}(V)$, where $V = \{(0, 0)\}$, is isomorphic to k.

(6) Consider the varieties $V, W \subset k^2$ given by

$$V = \{(x_1, x_2)\colon x_1^4 + x_2^4 = 1\} \qquad W = \{(y_1, y_2)\colon y_1^2 + y_2^2 = 1\}$$

and the morphism

$$\phi\colon k^2 \longrightarrow k^2$$
$$(x_1, x_2) \longmapsto (x_1^2, x_2^2).$$

Show that this gives a map of varieties $\phi\colon V \to W$.

(7) Recall from class the (partial) proof of the theorem which gives a correspondence between morphisms $\phi\colon V \to W$ and k-algebra homomorphisms $\psi\colon k[W] \to k[V]$.

(a) We had a map from $k[y_1, ..., y_m]$ to $k[V]$ given by composing $\phi^*\colon k[y_1, ..., y_m] \to k[x_1, ..., x_n]$ with the quotient map $q\colon k[x_1, ..., x_n] \to k[V]$ and we noticed that $\mathbb{I}(W)$ was mapped to 0. This allows us to define a homomorphism

$$k[y_1, ..., y_m]/\mathbb{I}(W) = k[W] \longrightarrow k[V].$$

(Since we know that $[0] \to [0]$ and the original map $q \circ \phi^*$ is a homomorphism.) Why is this map a k-algebra homomorphism? (This proves one half of the theorem.)

(b) Suppose the following diagram is commutative:

$$\begin{array}{ccc} k[y_1, ..., y_m] & \xrightarrow{\psi'} & k[x_1, ..., x_n] \\ \downarrow{\scriptstyle q_W} & & \downarrow{\scriptstyle q_V} \\ k[W] & \xrightarrow{\psi} & k[V] \end{array}$$

Here ψ, ψ' are k-algebra homomorphisms and q_V, q_W are quotient maps. Show that ψ' then gives a morphism $\phi\colon V \to W$ (where $\phi^* = \psi'$). (This proves the other half of the theorem, since we were given ψ and were able to construct ψ' from it.)