NAME: SOLUTIONS

Instructions: Check that your test has 12 pages, including this one and the blank one on the bottom. There are 10 problems on the exam. *Write neatly:* solutions deemed illegible will not be graded, so no credit will be given. You must show all work, justify all nonobvious parts of your work, and reference theorems or other facts you know from class or textbook in order to receive credit. Use full English sentences. This exam is closed book, closed notes. Calculators are not allowed.

PLEDGE: On my honor as a student, I have neither given nor received aid on this exam.

SIGNATURE: ________________

1. (12 points) ________
2. (9 points) ________
3. (9 points) ________
4. (5 points) ________
5. (5 points) ________
6. (5 points) ________
7. (9 points) ________
8. (5 points) ________
9. (8 points) ________
10. (8 points) ________

Total (out of 75): ________

1. (2 pts each) Answer each of the following questions by circling TRUE or FALSE. You do not need to justify you answer and no partial credit will be given.

(a) TRUE $\boxed{\text{FALSE}}$

The implication "If pigs can fly, then 1+1=3" is false. (see homework, section 1.1, problem 13(a))

(b) $\boxed{\text{TRUE}}$ FALSE

There are $P(n, m)$ one-to-one functions from a set with m elements to a set with n elements.
(example done in class, 9/19)

(c) TRUE $\boxed{\text{FALSE}}$

The sequence 15, 8, 1, -6, -13, ... is geometric. (see definitions 2, 3 on page 226)

(d) $\boxed{\text{TRUE}}$ FALSE

If A is an uncountable set and B is a countable set, then $A - B$ is uncountable.
(see homework, section 3.2, problem 33)

(e) $\boxed{\text{TRUE}}$ FALSE

There are $4C(13, 5)$ poker hands from one deck of cards which contain the same suit.
(example done in class, 9/26)

(f) TRUE $\boxed{\text{FALSE}}$

The conditional probability of an event E, given another event F, is $p(E|F) = p(E)p(F)$.
(see definition 3, page 366)

Part I: Things you've seen before

2. (3 pts each) State precise definitions of the following:

(a) The Cartesian product of sets A and B:

See definition 9, page 83.

(b) A surjective function:

See definition 7, page 100.

(c) The probability of an event E in a sample space S (do not assume equally likely outcomes):

See definition 2, page 364.

(Note that all these definitions were given in class as well)

3. (3 pts each) State the following:

(a) The Generalized Pigeonhole Principle:

See Theorem 2, page 314.

(b) The Binomial Theorem:

See Theorem 1, page 327.

(c) The formula for the number of r-combinations from a set with n elements when repetition of elements is allowed:

See Theorem 2, page 337.

(note that all these statements were given in class as well)

4. (5 pts) Prove that $\sqrt{2}$ is irrational.

See Example 21 on page 66 or class notes from 8/31.

5. (5 pts) Prove that the set of positive real numbers is uncountable.

See Example 20 on page 235 or class notes from 9/14.

6. (5 pts) Use induction to show $\displaystyle\sum_{i=1}^{2^n} \frac{1}{i} \geq 1 + \frac{n}{2}$.

See Example 6 on page 243 or class notes from 9/14.

Extra credit (2 pts): Why does this show that the harmonic series diverges?

The harmonic series is $\sum_{i=1}^{\infty} \frac{1}{i}$ but the above shows that this series eventually becomes larger than any number, and it thus diverges.

Part II: Things you haven't seen before

7. Consider $C = \{A \subseteq \mathbf{N} \mid A \text{ is finite} \vee \overline{A} \text{ is finite}\}$. Suppose A and B are elements of C.

(a) (3 pts) Show $\overline{A}$ is an element of C.

If A is finite, then $\bar{A}$ has a finite complement, i.e. $\bar{\bar{A}} = A$, so it is in C.

If A is infinite, then $\bar{A}$ is finite so it is in C.

(b) (3 pts) Show $A \cap B$ is an element of C.

If either or both of A and B is finite, so is $A \cap B$.

If they are both infinite, then $\bar{A}$ and $\bar{B}$ are both finite so $\bar{A} \cup \bar{B}$ is finite. But

$$\overline{A \cap B} = \bar{A} \cup \bar{B} \text{ so } \overline{A \cap B} \text{ is finite} \rightarrow A \cap B \in C$$

(c) (3 pts) Use parts (a) and (b) to show $A - B$ is an element of C.

By (a) $\bar{B} \in C$

By (b) $A \cap \bar{B} \in C$

$$\rightarrow A - B = A \cap \bar{B} \in C.$$

8. (5 points) Prove or disprove this cancellation property for functions:

$$\text{If } f \circ g = f \circ h, \text{ then } g = h.$$

Here is a counterexample:

Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, $C = \{c_1\}$.

Let

$g: A \longrightarrow B$	$h: A \longrightarrow B$	$f: B \longrightarrow C$
$a_1 \longmapsto b_1$	$a_1 \longmapsto b_1$	$b_1 \longmapsto c_1$
$a_2 \longmapsto b_2$	$a_2 \longmapsto b_1$	$b_2 \longmapsto c_1$
$a_3 \longmapsto b_2$	$a_3 \longmapsto b_2$	

Then

$$(f \circ g)(a_1) = (f \circ h)(a_1) = c_1$$
$$(f \circ g)(a_2) = (f \circ h)(a_2) = c_1$$
$$(f \circ g)(a_3) = (f \circ h)(a_3) = c_1$$

So $f \circ g = f \circ h$.

But $g \neq h$.

9. (8 pts) Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers. (Hint: Any integer can be written as a power of 2 times an odd integer.)

Let the $n+1$ integers be $a_1, a_2, \ldots, a_{n+1}$ and write each as a power of 2 times an odd integer:

$$a_i = 2^{b_i} c_i, \quad i = 1, 2, \ldots, n+1, \quad c_i \text{ odd}.$$

All c_i are less than $2n$ (since otherwise a_i would be $> 2n$), and positive (by assumption, all a_i are positive).

But there are only n odd integers greater than 0 and smaller than $2n$, so by Pigeonhole Principle,

$$\exists\, j, k \text{ s.t. } c_j = c_k.$$

Let $c = c_j = c_k$.

Then we have $a_j = 2^{b_j} c$, $a_k = 2^{b_k} c$.

The smaller of these divides the larger one.

10. (8 pts) If each performance of an experiment has only two outcomes, such as a coin toss, the performance is called a *Bernoulli trial.* The two outcomes are usually called *success* and *failure.* Let the probability of a success be p and the probability of a failure be $q = 1 - p$. Show that the probability of exactly k successes in n independent Bernoulli trials is

$$C(n,k)p^k q^{n-k}.$$

The outcome of n independent Bernoulli trials can be thought of as a n-tuple

$$(a_1, a_2, \ldots, a_n)$$

where $a_i = S$ (success) or

$a_i = F$ (failure).

Since trials are independent, the probability of k S's (and $n-k$ F's, consequently) is the product of the individual probabilities, i.e.

$$p^k q^{n-k}.$$

But the k S's could appear in the n-tuple in any of

$C(n,k)$ ways, so by product rule the total probability is

$$C(n,k) p^k q^{n-k}.$$