MATH 475 KNOT THEORY — FINAL EXAM
May 2, 2005

This exam is due Monday, May 9, in my mailbox by 11 am. You should work alone, and may use notes, homeworks (and everything proved there), and Adams' book.

1. (10 pts) Given that $\Big\langle$  $\Big\rangle = -A^4 - A^{-4}$ and $\Big\langle$  $\Big\rangle = A^7 - A^3 - A^{-5}$, find the X-polynomial of the knot in Figure 1.

2. (10 pts) A link is called *Brunnian* if it is nontrivial, yet removing any one of its components leaves a trivial link. Figure 2 shows a Brunnian link of three components. Find a Brunnian link of four components.

3. (a) (5 pts) Draw the braid corresponding to the word $w = (\sigma_1^{-1}\sigma_2\sigma_3^{-1})^2\sigma_3^3$.

(b) (5 pts) Draw the closure of this braid and identify it as one of the knots in the table from the back of the book.

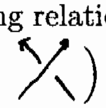
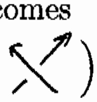
(c) (5 pts) Draw the braid corresponding to the inverse of w.

4. (a) (5 pts) Give an example of a space which fails to be a 3-manifold at 3 points.

(b) (5 pts) Read the definition of the *double* of a manifold M on page 252. What manifold is S^3 a double of?

(c) (5 pts) In a short paragraph, describe how lens spaces arise from Dehn surgery.

5. (10 pts) Suppose we make a substitution $s = t^{-1/2} - t^{1/2}$ in the definition of the Alexander polynomial so that its defining relation becomes

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = s\Delta\Big($  $\Big).$

Show that every coefficient of the Alexander polynomial defined this way is a finite type n invariant for some n.

6. (10 pts) Exercise 2.30, page 54.

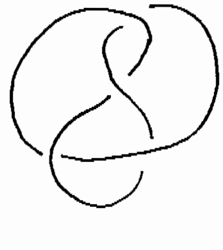


Figure 1.

Figure 2.

FINAL EXAM SOLUTIONS

(1) The writhe of the given knot is 0,

therefore $X(K) = \langle K \rangle$.

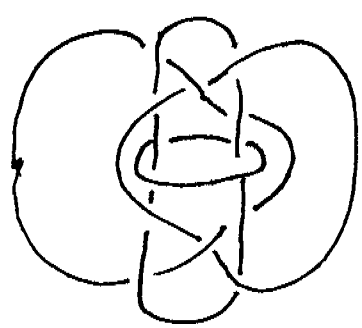
$X \langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

this is the mirror image of what's given, so

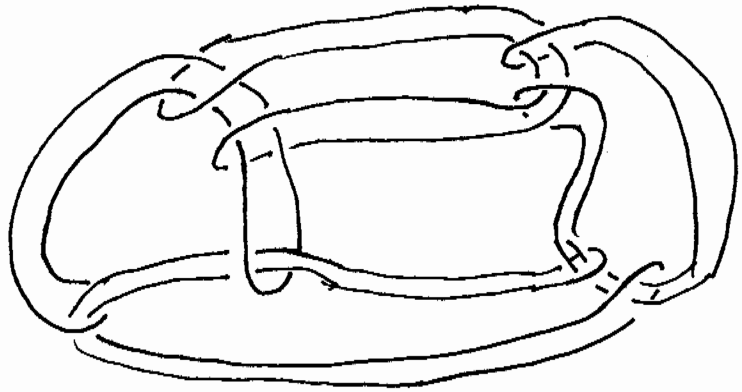
$\langle$  $\rangle = A^{-7} - A^{-3} - A^{5}$

$$= A^{8} - A^{4} + 1 - A^{4} + A^{-8}$$

(2) The solution is



Another one is



(This might be isotopic to the first one. This one is nice because it generalizes to Brunnian links of any number of components.)

3 (a)

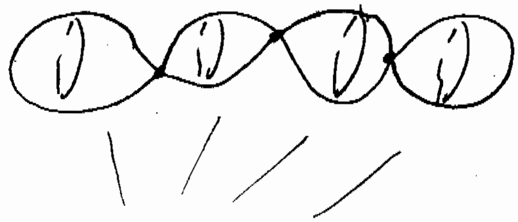
(b) σ_1

(c)

$= \sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^{-1}\sigma_1$

4

(a)



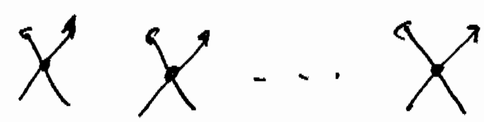
4 solid spheres touching at 3 points.

(b) S^3 is a double of a 3-ball

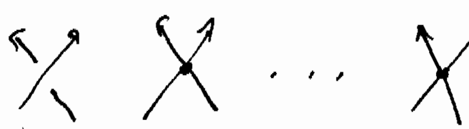
(c) Dehn surgery is the process of gluing a torus (solid) into a drilled-out tubular neighborhood of a knot in S^3 along a (p,q)-torus knot. One gets lens spaces this way in particular.

5

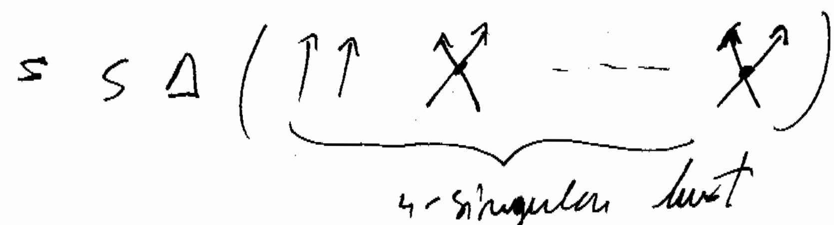
Vassiliev skein rel'n

$\Delta\Big($  $\Big) =$

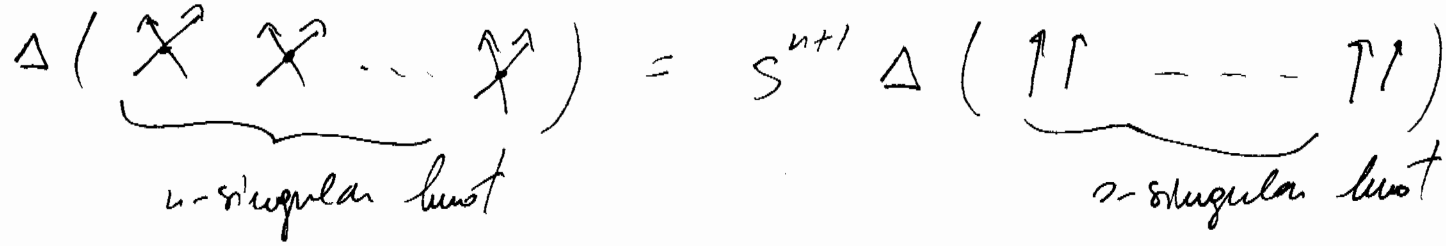
(n+1)-singular knot

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) =$

n-singular knots



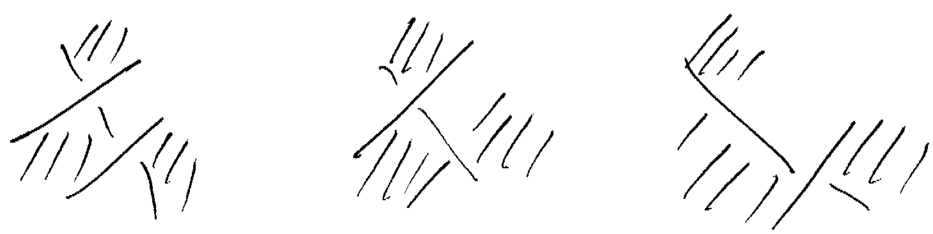
Do this n more times and get



Now $\Delta(\uparrow\uparrow \cdots \uparrow\uparrow)$ is some polynomial in positive powers of s so the resulting polynomial has s^{n+1} as its lowest power. This means that the coefficient of s^n is a ~~##~~ 0 and is hence a type n invariant.

(b)

If a link has an alternating projection, looking at 2 consecutive crossings gives an overcrossing followed by undercrossing (or vice versa) and these will border the same shaded region so both crossings are either + or − !



+, − two +'s two −'s

It follows that all crossings must have the same sign.

Conversely, can run this argument backward.