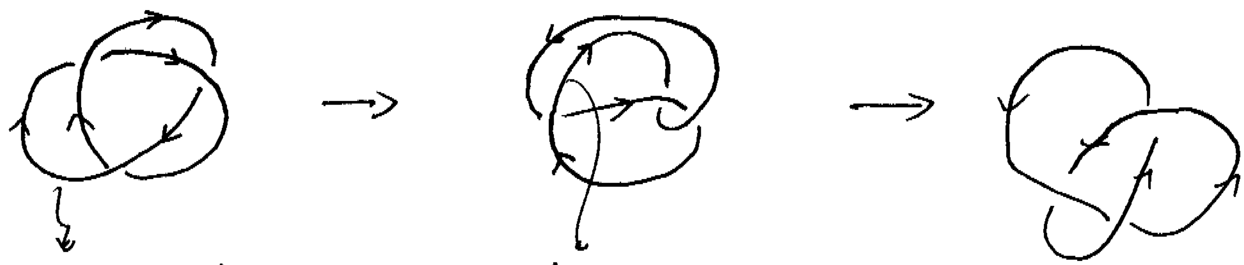
HW 2 SOLUTIONS

2/4/05

1. Find an isotopy which reverses the orientation of the trefoil:

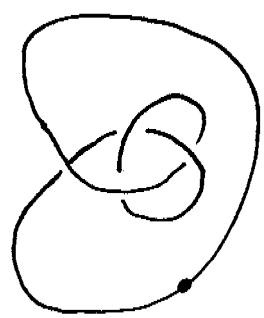


flip this strand over

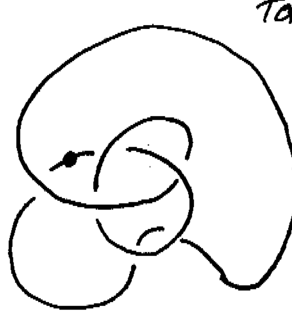
flip this strand over

$\longrightarrow$ planar isotopy (rotation)

2. Exercise 1.10. (dot (•) indicates where the move is taking place)

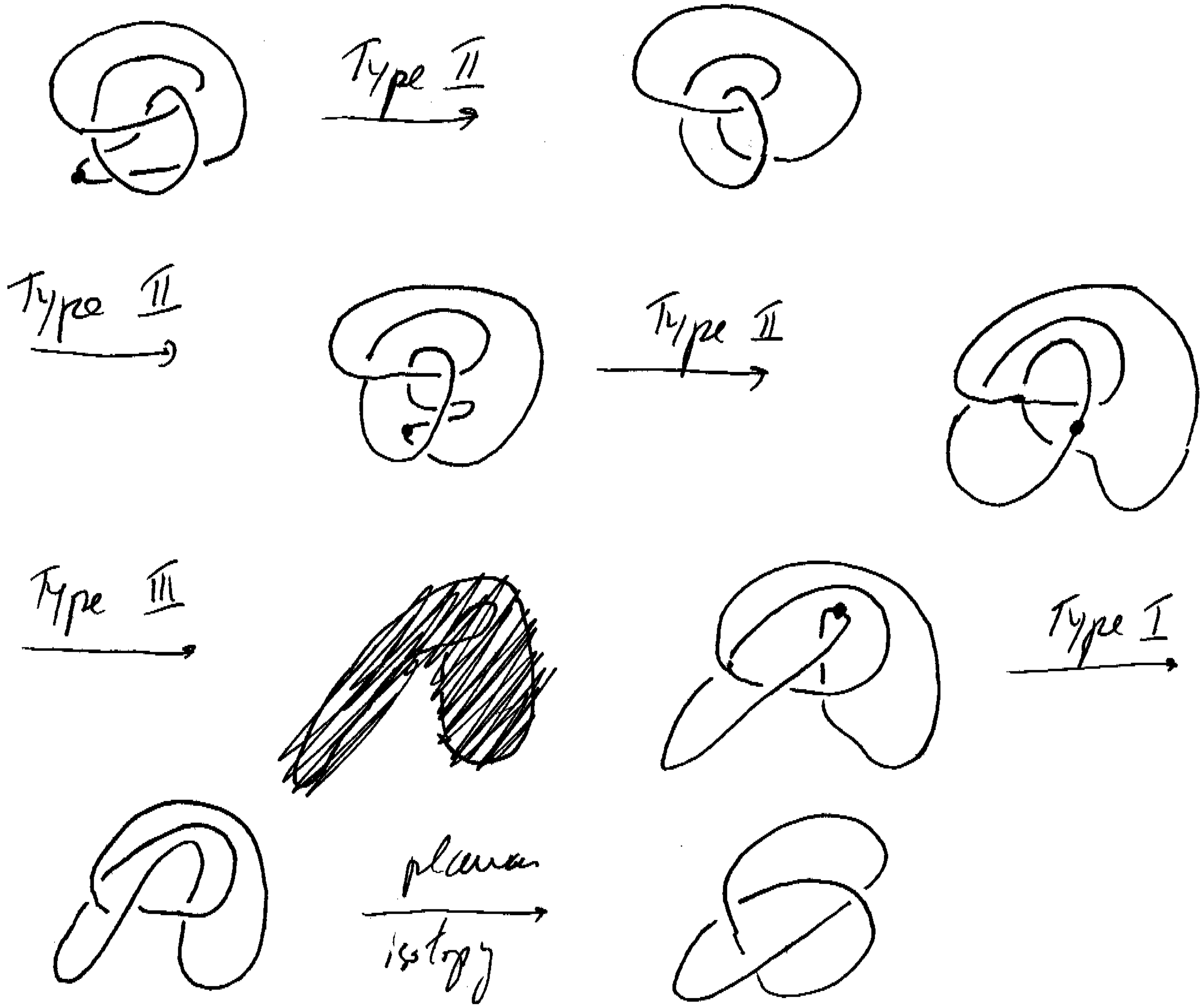


$\xrightarrow{\text{Type II}}$

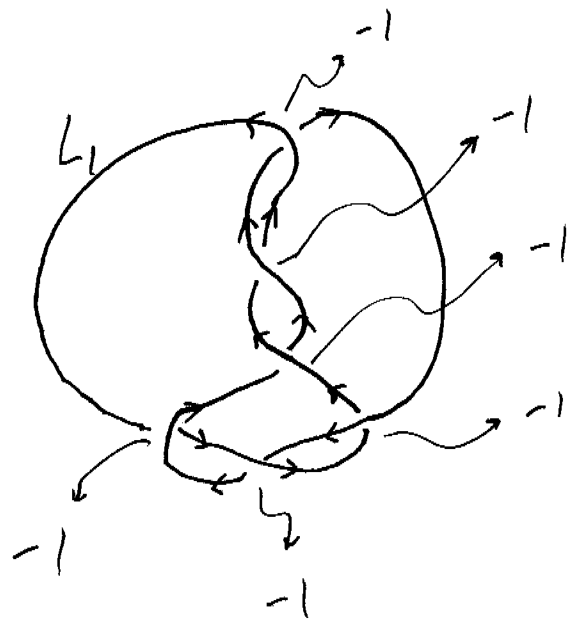


$\xrightarrow{\text{Type III}}$

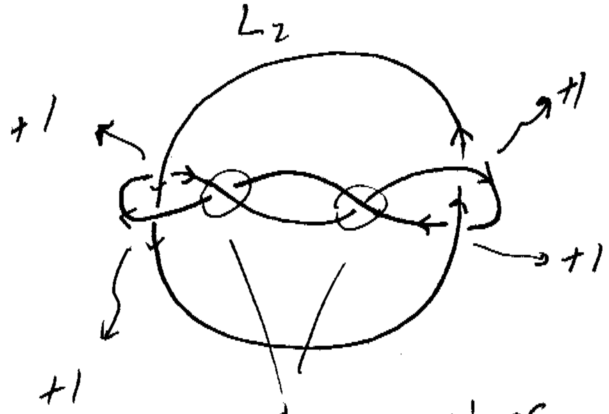
$\longrightarrow$ planar isotopy



3 Exercise 1.17.



$$lk(L_1) = \frac{1}{2}\,|\,6\cdot(-1)\,|$$
$$= 3$$



$$lk(L_2) = \frac{1}{2}|4 \cdot 1| = 2$$

Since $2 \neq 3$, $L_1 = L_2$.

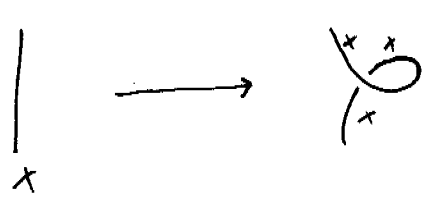
(4) <u>Exercise 1.28*</u>

(a) Label the strands of a knot by numbers 0, 1, 2, 3, 4 such that, at each crossing,

$$x + y \equiv 2z \pmod 5, \quad x, y, z \in \{0, 1, 2, 3, 4\}$$

Let's show this kind of labeling is preserved under Reidemeister moves.

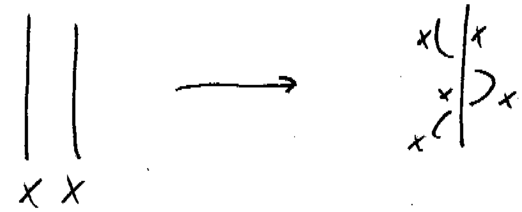
<u>Type I:</u>



Here $x = y = z$, and $x + x = 2x \equiv 2x \pmod 5$

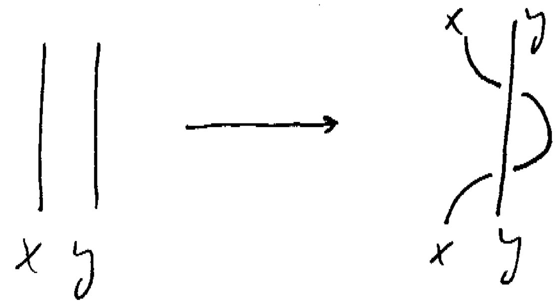
(because 5 divides 0)

Type II:



Still $x+x \equiv 2x \pmod 5$

or

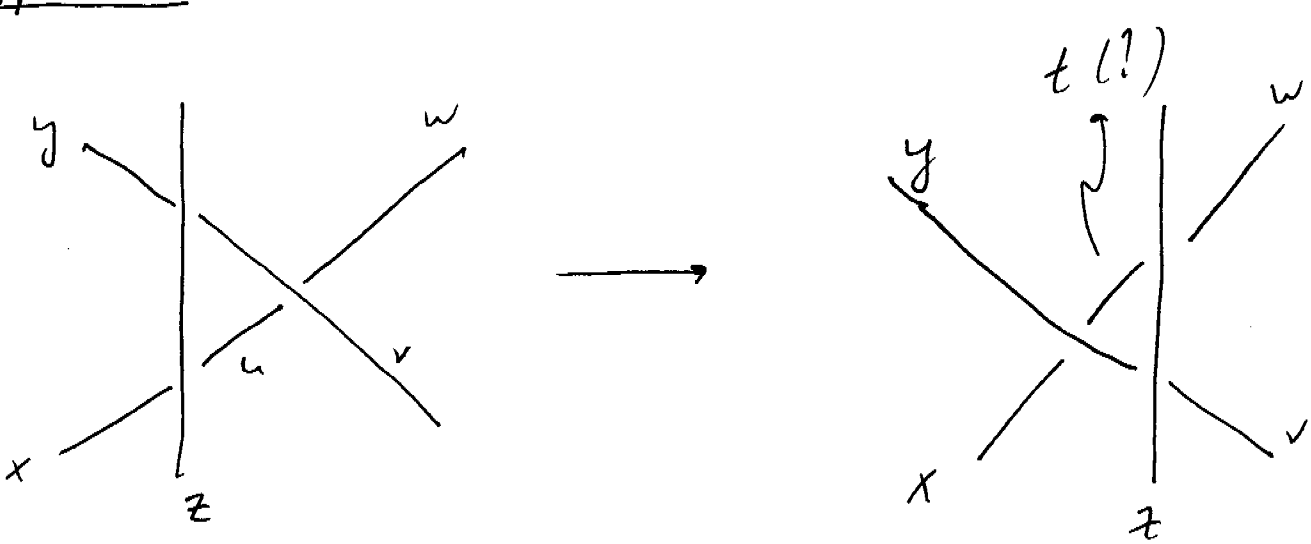


label this by $2y-x \pmod 5$.
This is because:
we're looking for some u such that $x+u = 2y \pmod 5$, i.e.

y is the overstrand

we're looking for $u \equiv 2y-x \pmod 5$.
So whatever x, y are, choose u like this and labeling scheme is satisfied.

Type III:



In the left picture, we are given some labeling satisfying

(i) $y + v = 2z \pmod 5$

(ii) $x + u = 2z \pmod 5$

(iii) $u + w = 2v \pmod 5$

The <u>question is</u>: Given the above conditions, can we label the right picture, i.e. can we find t, such that

(a) $x + t \equiv 2y \pmod 5$

(b) $w + t \equiv 2z \pmod 5$

(c) $y + v \equiv 2z \pmod 5$?

Note (i) = (c) so can disregard (i).

(a) $\iff$ $t = 2y - x \pmod 5$

(b) $\iff$ $t = 2z - w \pmod 5$

So solution exists provided

$$2y - x \equiv 2z - w \pmod 5 \quad (*)$$

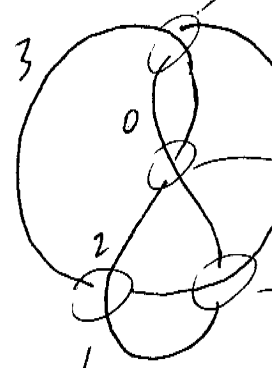
But (iii) − (ii) $\Rightarrow$ $w - x \equiv 2v - 2z \pmod 5$

$\overset{(c)}{\equiv} 2(2z - y) - 2z \pmod 5$

$\equiv 2z - 2y \pmod 5$

Rearranging this gives exactly ⊛ as desired. (b)

Now apply this to figure-8.



$1 + 0 \equiv 2 \cdot 3 \pmod{5}$ ✓

$2 + 3 \equiv 2 \cdot 0 \pmod{5}$ ✓

$0 + 2 \equiv 2 \cdot 1 \pmod{5}$ ✓

$3 + 1 \equiv 2 \cdot 2 \pmod{5}$ ✓

So figure-8 is 4-colorable. Since unknot isn't, we get

figure-8 $\neq$ unknot

(b) Same as above:

Require $x + y \equiv 2z \pmod{3}$,

where z is the overstrand and

$x, y, z \in \{0, 1, 2\}$.