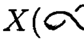
MATH 475 KNOT THEORY — MIDTERM EXAM
March 14, 2005

This exam is due Monday, March 21, in class or in my mailbox my 11 am. You should work alone, and may use notes, homeworks (and everything proved there), and Adams' book.

1. (10 pts) Given a diagram for an oriented knot K, show that there exists a 2-component link L with $w(K) = lk(L)$, where K is one component of L.

2. (5 pts) Given a choice between computing $\langle$$\rangle$ or $X($$)$, which problem would you choose and what would your answer be? Explain.

3. (10 pts) Calculate the genus of $6_3 \# 7_6$.

4. Let K be the knot shown in Figure 1. In the notation of the knot tables, the Jones polynomial of K is $\{-3\}(-1\ 3\ \ -3\ 4\ \ -4\ 3\ \ -2\ 1)$.

(a) (5 pts) Find $X(K)$.

(b) (5 pts) Sketch the mirror image K^* of K and find $X(K^*)$.

(c) (10 pts) What do (a) and (b) tell you about K and K^*? Is it ever possible to distinguish between a knot and its mirror image using, say, the crossing number? Why or why not?

5. (a) (5 pts) Compute the linking number of the oriented link in Figure 2(a).

(b) (5 pts) Suppose we add vertices a, b, c, d, e, f as shown in Figure 2(b). We can then add various edges so as to obtain an embedding of K_6 in $\mathbb{R}^3$. Is it possible to add the edges in such a way that (abc, def) is the only pair of linked triangles? Justify your answer.

6. (a) (10 pts) Are the knots K and K' is Figure 3 pass equivalent? Justify your answer.

(b) (10 pts) Are the knots K and K' is Figure 4 pass equivalent? Justify your answer.

7. (10 pts) Show by a sequence of pictures that if K is the trefoil, $K \# K^*$ is pass equivalent to the unknot.

8. (15 pts) Exercise 2.13, page 46. You'll have to read Section 2.3 and learn what a tangle is and how to associate a rational number to a tangle before you can do this problem. Then use the statement right before the exercise to do the exercise.

Midterm Figures

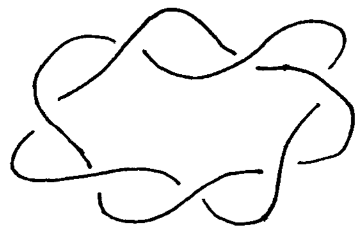


Figure 1.

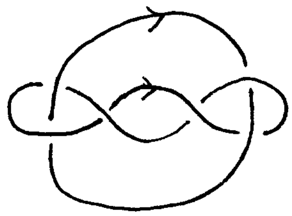


Figure 2(a)

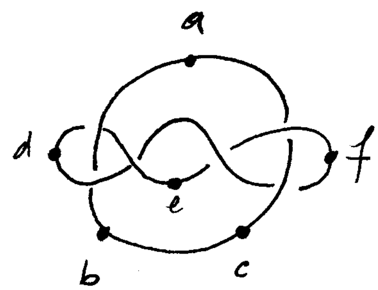


Figure 2(b)

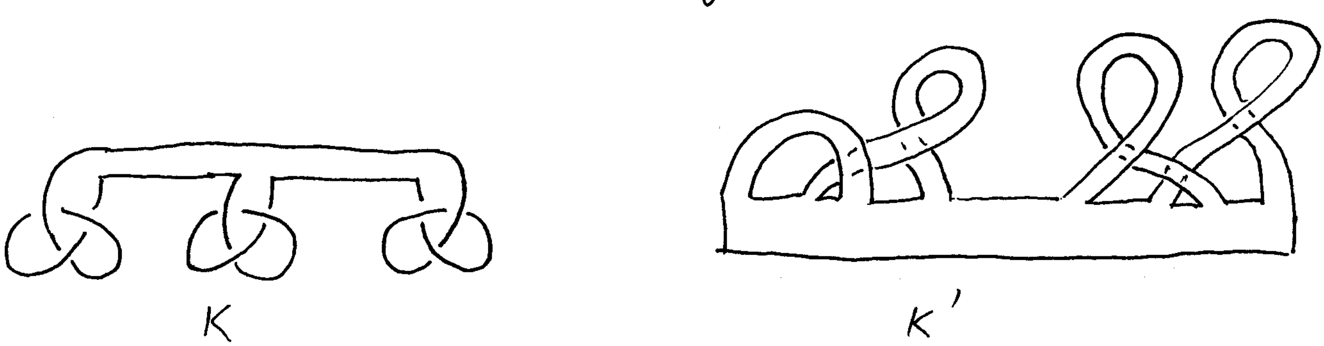


Figure 3

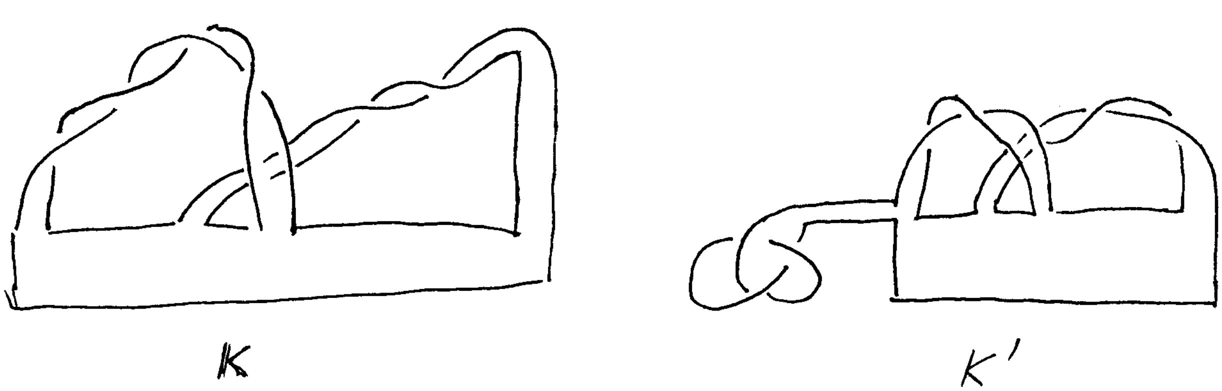
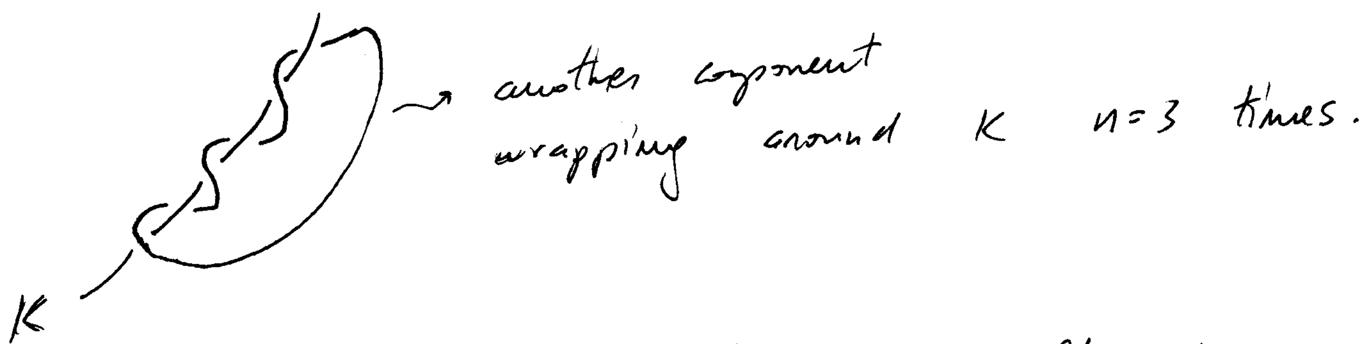


Figure 4

MIDTERM SOLUTIONS

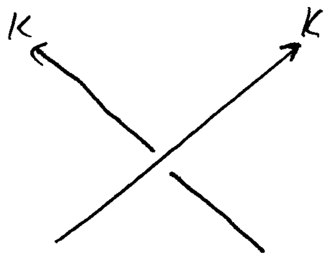
1 Given K with $|w(K)| = n$, choose the other component of L so that it wraps around a small part of K n times:

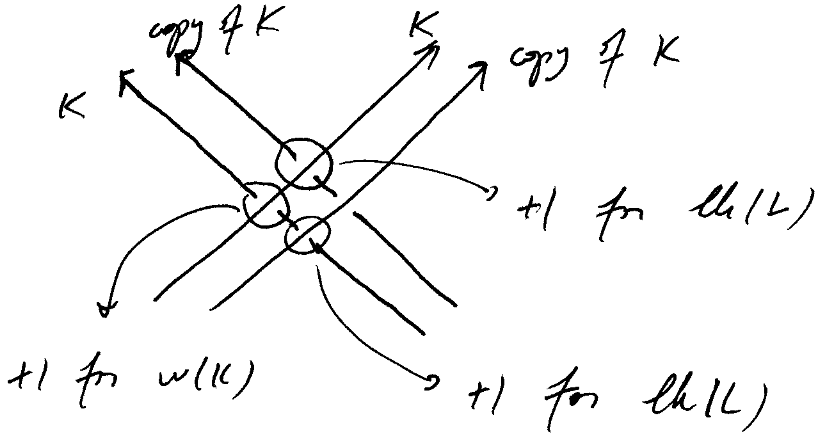


It is immediate that $|w(K)| = |lk\ L|$.
To get the right signs, i.e. to get the right answer without absolute values, choose the correct orientation on the other component.

Another way to do this would be to choose the other component to be just a slightly displaced copy of K with same orientation. Then for each crossing of K there would be 2 crossings between K and the other component. But the factor of $\frac{1}{2}$

for the linking # would finally give $w(K) = lk(L)$. E.g. we would have





(2) Since $X(\cdot)$ is an invariant, but $\langle \cdot \rangle$ isn't, we immediately get

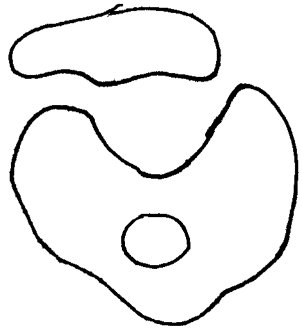
$$X(\infty) = X(\bigcirc) = 1,$$

while computing $\langle \infty \rangle$ is harder.

(3) Use $g(K \# L) = g(K) + g(L)$ and compute genera separately

$\boxed{6_3}$

yields

 $\rightarrow$ 3 Seifert discs.

Since 6_3 has 6 crossings,

$$\chi(6_3) = s - c = 3 - 6 = -3$$

and $\chi(6_3) = -3 = 2 - 2g(6_3) - 1$

$$\Rightarrow g(6_3) = 2.$$

$\boxed{7_6}$ Get 4 Seifert discs, so

$$\chi(7_6) = 4 - 7 = -3$$

$$\Rightarrow g(7_6) = 2.$$

<u>Note</u> that both 6_3 and 7_6 are alternating, so we are really getting <u>minimal</u> <u>genus</u>.

Thus

$$g(6_3 \# 7_6) = 2 + 2 = 4.$$

4 4

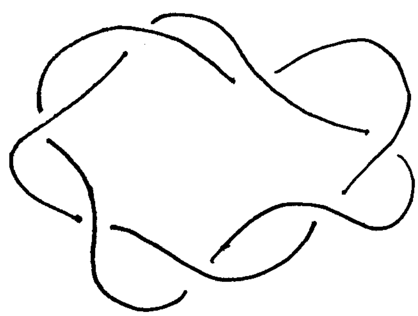
(a) Jones polynomial is

$$-t^{-3}+3t^{-2}-3t^{-1}+4-4t+3t^{2}-2t^{3}+t^{4}$$

X-polynomial is obtained by substituting $t = A^{-4}$:

$$X(K) = -A^{12}+3A^{8}-3A^{4}+4-4A^{-4}+3A^{-8}-2A^{-12}+A^{-16}$$

(b)



$= K^*$

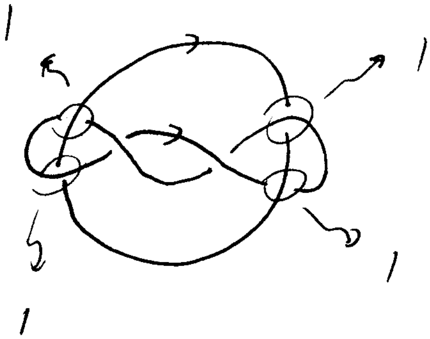
$$X(K^*)(A) = X(K)(A^{-1})$$

$$= -A^{-12}+3A^{-8}-3A^{-4}+4-4A^{4}+3A^{8}-2A^{12}+A^{16}$$

(c) Since $X(K)$ is not palindromic, we know $K \neq K^*$. However any knot and its mirror image have the same crossing number. Thus crossing number says nothing about a knot and its mirror image.

(5) (5)

(a)



$lk(L) = \frac{1}{2} 4 = 2$

(b) It is not possible because we know the sum of all linking numbers of pairs of triangles in K_6 must be odd. Since the above linking # is even, there has to be at least one more pair of linked triangles with odd linking number.

(6)

(a) Knot K is (trefoil)#(trefoil)#(trefoil)*

From class we know trefoil and trefoil* are pass-equivalent (p.e.), and trefoil # trefoil is p.e. to unknot.

Thus

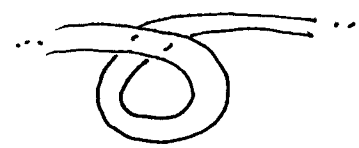
(trefoil) # (trefoil) # (trefoil)* is p.e. to

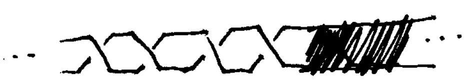
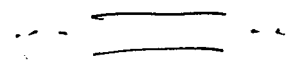
$(\text{trefoil}) \# \underbrace{(\text{trefoil}) \# (\text{trefoil})}_{\text{unknot}}$, which is p.e. to <u>trefoil</u>.

From class, we also know that K' is p.e. to trefoil since left half is p.e. to unknot and right half is p.e. to trefoil so K' is p.e. to (unknot) # (trefoil) = trefoil.

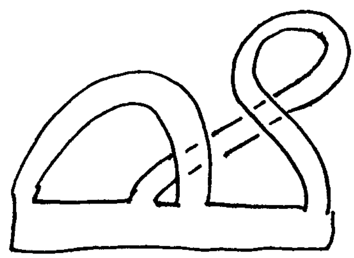
Hence K and K' are p.e.

(b) Recall from class that

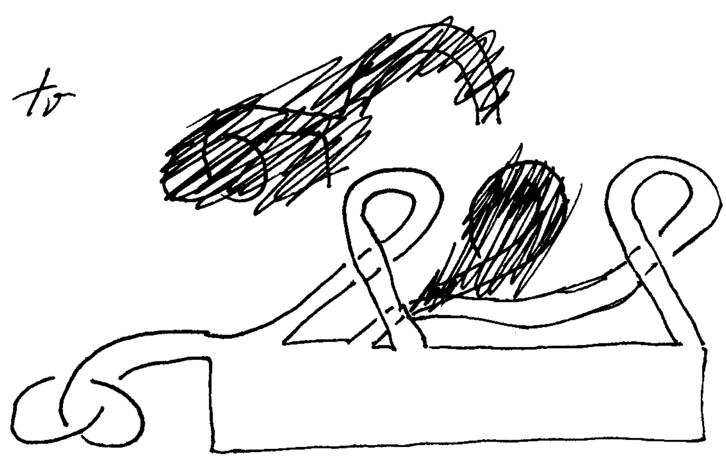
 is p.e. to 

 is p.e. to 

Thus

K is p.e. to , which is p.e. to the unknot.

K' is p.e. to

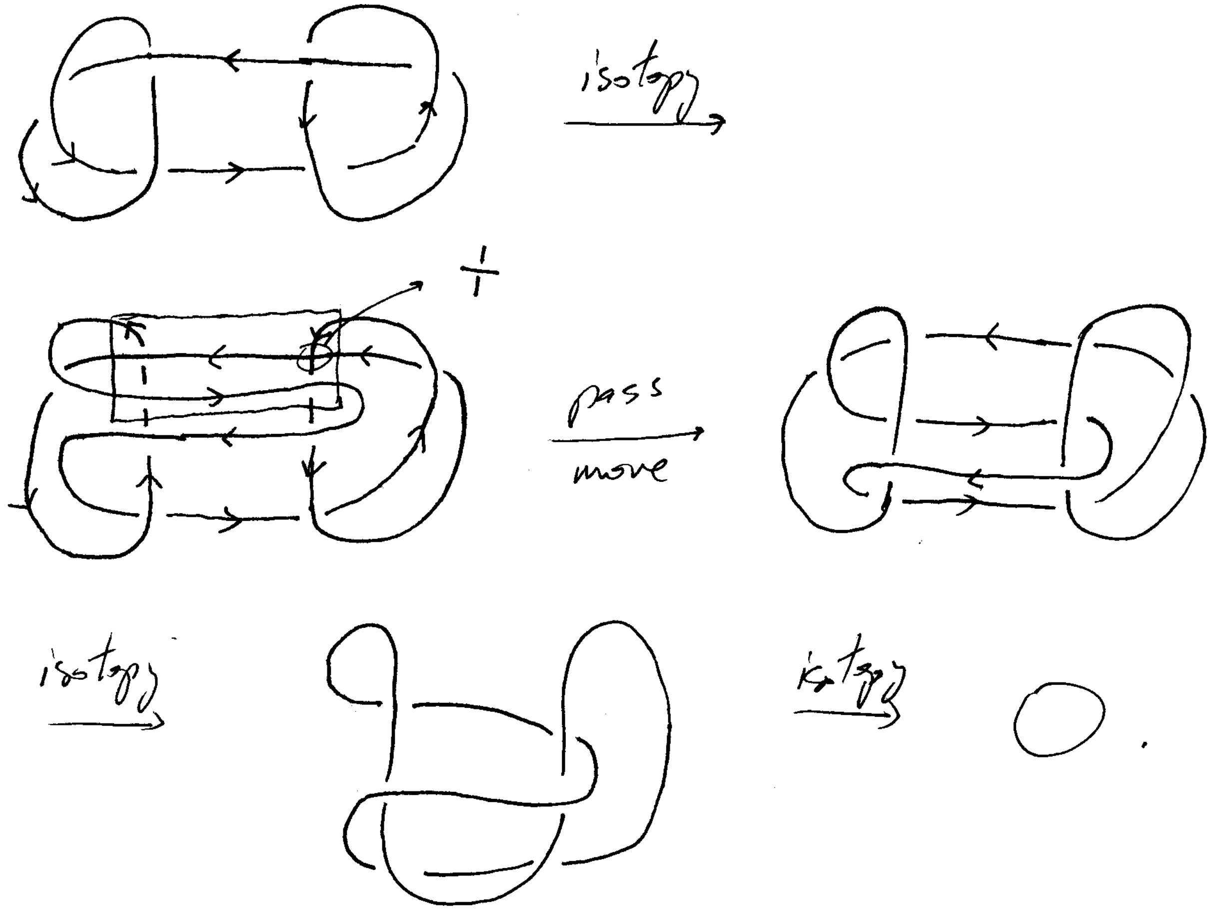


(Sorry about the messy picture)

which is p.c. to (trefoil) # (trefoil),
which is p.c. to the unknot.

Thus K is p.c. to K'.

(7)



Note: There are various ways to do this problem.

(8) Tangle 1 is $-2\ -2\ -2\ -2\ -2$ w/ fraction $-29/12$

— " — $2\ -3\ 2$ — " — $8/5$

— " — $-4\ 1\ 2$ — " — $10/3$

— " — $1\ 1\ 1\ 1\ 1$ — " — $8/5$

So Tangles 2 and 4 are same.