Manifolds, K-theory and the Calculus of Functors

Michael Ching

Amherst College

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joint with Greg Arone
Overview

1. Calculus of Functors:
   Taylor towers of functors $\text{Top}_\ast \rightarrow \text{Sp}$

2. Manifolds:
   a category of ‘pointed framed manifolds’

3. K-Theory:
   Waldhausen's functor $A : \text{Top}_\ast \rightarrow \text{Sp}$
We study functors $F : \text{Top}_* \to \text{Sp}$ that preserve weak equivalences (and filtered homotopy colimits):

- $F$ has a Taylor tower expanded at $*$

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to F(*)$$

- and derivatives $\partial_n F$ (a spectrum with $\Sigma_n$-action) such that

$$D_n F(X) := \text{hofib}(P_n F(X) \to P_{n-1} F(X)) \simeq [\partial_n F \wedge (\Sigma \infty X)^\wedge n]_{h\Sigma_n}$$

**Main Question:** What structure does $\partial_\ast F = \{\partial_n F\}_{n \geq 1}$ possess and how can $P_n F$ be recovered from this?
Divided power right Lie-modules

**Theorem (Arone-C)**

For $F : \text{Top}_* \to \text{Sp}$ and $n = n_1 + \cdots + n_k$, there are (compatible) structure maps

$$\text{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_k} I, \partial_n F)_{h\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}$$

from which the Taylor tower of $F$ can be recovered (by a cobar construction).

Note: $\partial_* I = K(\text{Com})$: Koszul dual of commutative operad
Example: Derivatives of $A(X)$

Theorem (Goodwillie)

$$\partial_n A \simeq \partial_n (\sum \infty L)^{hS^1}$$

where $L = \text{Hom}(S^1_+, -)$ is the free loop space functor.

Goal: Describe the divided power $\partial_* I$-module structure on $\partial_* A$ that determines the Taylor tower of $A$. 
Additional structure on $\partial_* F$ for some $F$

The stable little $d$-discs operad $E_d$ has terms:

$$E_d(n) := \Sigma^\infty \{ \text{‘standard’ embeddings } \coprod_n D^d \to D^d \} +$$

There are operad maps

$$E_d \xrightarrow{f} \Com; \quad \partial_* I = \K(\Com) \xrightarrow{K(f)} \K(E_d).$$

By pullback along $K(f)$, a (divided power) $\K(E_d)$-module structure on $\partial_* F$ determines a divided power $\partial_* I$-module structure on $\partial_* F$.

The terms of the operad $\K(E_d)$ are finite free $\Sigma_n$-spectra, so ‘divided power’ adds nothing to a $\K(E_d)$-module.

**Question:** When does the Taylor tower of $F$ arise from a $\K(E_d)$-module structure on $\partial_* F$?
Pointed framed manifolds

Definition

A *pointed framed $d$-manifold* consists of:

- a finite based CW-complex $(X, x_0)$;
- the structure of a framed smooth $d$-dimensional manifold (with $\partial$) on $X - \{x_0\}$.

A *pointed framed embedding* $f : (X, x_0) \rightarrow (Y, y_0)$ consists of:

- a basepoint-preserving map $f : X \rightarrow Y$ such that the restriction of $f$ to

$$f^{-1}(Y - \{y_0\}) \subseteq X - \{x_0\}$$

is a ‘framed embedding’ ($Df$ is a locally constant positive multiple of the identity matrix).

These form a category $f\text{Mfld}_d^*$. (Compare to the ‘zero-pointed manifolds’ of Ayala-Francis.)
Examples of pointed framed $d$-manifolds and embeddings

- $M_+$ when $M$ is a framed compact $d$-manifold (e.g. $S^1_+$ when $d = 1$)
- $S^d = D^d / S^{d-1}$
- $M/A$ where $A \subseteq \partial M$ is a subcomplex
- ‘standard’ embeddings $\bigvee_n D^d_+ \to D^d_+$
- Pontryagin-Thom collapse maps $S^d \to \bigvee_n S^d$ associated to standard embeddings of discs
Theorem (Arone-C)

$F : \text{Top}_\ast \to \text{Sp}$ reduced, polynomial. The following are equivalent:

1. $\partial_\ast F$ has a $K(E_d)$-module structure (from which the Taylor tower can be recovered);

2. $F$ is the left Kan extension along $\text{fMfld}^d_\ast \to \text{Top}_\ast$ of a functor $G : \text{fMfld}^d_\ast \to \text{Sp}$;

3. $F(X) \simeq N \wedge_{E_d} \Sigma^\infty X^{\wedge_\ast}$ for some $E_d$-comodule $N$ (which is then Koszul dual to $\partial_\ast F$)

Also, 2 $\Rightarrow$ 1 for non-polynomial $F$. 
Theorem (Arone-Blumberg-C)

The derivatives of $A$ have the structure of a $K(E_?)$-module.

Proof.

Look at $p$-complete setting and then use arithmetic square.

(BCCGHM): $\partial_* A_p = \partial_* TC_p$ and there are homotopy pullbacks

\[
\begin{array}{ccc}
TC_p & \rightarrow & \Sigma(\Sigma^\infty L_p)_{hS^1} \\
\downarrow & & \downarrow Tr \\
\Sigma^\infty L_p & \rightarrow & \Sigma^\infty L_p
\end{array}
\]

By previous Theorem, $\partial_*(\Sigma^\infty L_p)$ is a $K(E_1)$-module.

**BUT**: the $p^{\text{th}}$ power map $\Delta_p : S^1 \rightarrow S^1$ is NOT AN EMBEDDING!
Structure on $\partial_* A$

**Theorem (Arone-Blumberg-C)**

The derivatives of $A$ have the structure of a $K(E_?)$-module $K(E_3)$-module.

**Proof (cont.)**

There is a model for $\Delta_p$ that is a framed embedding of framed 3-manifolds (solid tori):

So $1 - \Delta_p$ is a map of $K(E_3)$-modules $\partial_*(\Sigma^\infty L_p) \to \partial_*(\Sigma^\infty L_p)$. 

\[\square\]
Consequences for the Taylor tower of $A$

- $P_n A$ is the left Kan extension along $f \text{Mfld}^3_* \to \text{Top}_*$ of a functor $G_n : f \text{Mfld}^3_* \to \text{Sp}$

  (We don’t know that $A$ itself is such a Kan extension.)

- We can write

  $$P_n A(X) \simeq A(*) \times (N_n \wedge_{E_3} \Sigma^\infty X^\wedge*)$$

  for the $E_3$-comodule $N_n$ that is Koszul dual to the $K(E_3)$-module $\partial_{\leq n} A$. 