On the Topology of Diagonal Arrangements

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Diagonal Arrangements

The classical configuration space of \( n \) points in \( X \) is

\[
F(X, n) = \{(x_1, \ldots, x_n) \mid x_i \neq x_j, i \neq j\}
\]

This generalizes to the “no \( d + 1\)-equal” configurations

\[
\Delta^d(X, n) = \{(x_1, \ldots, x_n) \mid \text{at most } d \text{ equal entries}\}
\]

where \( \Delta^1(X, n) = F(X, n) \).

If one defines the \( d \)-equal configuration space

\[
\Delta_d(X, n) = \{(x_1, \ldots, x_n) \mid x_{i_1} = \cdots = x_{i_d} \text{ for some sequence } i_1 < i_2 < \ldots < i_d\}
\]

Then

\[
\Delta^d(X, n) = X^n - \Delta_{d+1}(X, n)
\]
Example $\Delta_n(\mathbb{R}^k, n)$ is the diagonal in $(\mathbb{R}^k)^n$.
$\Delta^{n-1}(\mathbb{R}^k, n)$ is the complement in $(\mathbb{R}^k)^n$ of the thin diagonal

$$\Delta^{n-1}(\mathbb{R}^k, n) \simeq S^{k(n-1)-1}$$

The spaces $\Delta_d(X, n)$ are special cases of so-called diagonal arrangements.

If $K$ is a simplicial complex on the index set $[m]$ (vertices), we define

$$\Delta_K(X) = \bigcup_{\sigma \in K} \Delta_\sigma(X)$$

where

$$\Delta_\sigma(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_{i_1} = \ldots = x_{i_k} \text{ for } \{i_1, \ldots, i_k\} = [m] - \sigma\}$$

If $K$ is the $n-2$-skeleton of the $n$-simplex, then $\Delta_K(X)$ is the braid arrangement, and when $K$ is the $n-d$-skeleton, $\Delta_K(X)$ is our $\Delta_d(X)$. 
In this talk we will only focus on $\Delta_d(X)$ and their complements $\Delta^d(X)$, as well as their unordered analogs

$$B_d(X, n) := \Delta_d(X, n)/\mathcal{S}_n \quad , \quad B^d(X, n) := \Delta^d(X, n)/\mathcal{S}_n$$

$\mathcal{S}_n$ is the $n$-th symmetric group acting by permutations.

We have a descending filtration

$$B_1(X, n) = SP^nX \supset B_2(X, n) \supset \cdots \supset B_n(X, n) = X$$

and an ascending one

$$B^1(X, n) = B(X, n) \subset B^2(X, n) \subset \cdots \subset B^n(X, n) = SP^nX$$

where $B(X, n)$ is the braid space

$$B(X, n) = \{ S \subset X \mid |S| = n \}$$

Similar filtration for the ordered case.
- These spaces appear in work of F. Cohen and E. Lusk (1974) who compute the homology of $\Delta^{q+1}(\mathbb{R}^n, p)$ with $\mathbb{Z}_p$ coefficients. This is in connection with a generalization of the classical **Borsuk-Ulam Theorem** which consists in finding conditions on $X$ so that for (any) map $f : X \rightarrow \mathbb{R}^n$, $X$ a free $\mathbb{Z}_p$-space, the set

$$\{x \in X \mid \exists \ i_1 < i_2 < \ldots < i_q \text{ so that } f(\sigma^{i_1}x) = \ldots = f(\sigma^{i_q}x)\}$$

is non-empty, or better, has some non-trivial covering dimension.

- The no $d$-equal configurations also appear in work of Bjorner-Welker who compute the homology of $\Delta^d(\mathbb{R}, k)$ and $\Delta^d(\mathbb{C}, k)$ as interesting examples of complements of subspace arrangements. The homology is **torsion free**.

- Most recently, the homology and cohomology of $\Delta^d(\mathbb{R}^m, k)$ for all $m$ has been described by Turchin and Dobrinskaya with a view towards application to spaces of **non-$k$ equal immersions**.
Fundamental Group

Assume throughout $X$ based connected CW complex.

It is well-known that

$$\pi_1(\text{SP}^n(X)) \cong H_1(X, \mathbb{Z})$$

This corresponds to the case $d = 1$ in $B_d(X, n)$.

Interestingly when $d > n/2$, $B_d(X, n) \cong X \times \text{SP}^{n-d}(X)$, so the space is obvious.

**Theorem** (K, Taamallah)

Suppose that $1 \leq d \leq \frac{n}{2}$. Then

$$\pi_1(B_d(X, n)) \cong H_1(X, \mathbb{Z})$$

The proof consists in lifting a loop in $B_d(X, n)$ to its $\Sigma_n$-branched covering $\Delta_d(X, n)$ so that for given two loops, the lifts can be chosen to commute upstairs (P.A. Smith).
The spaces $B^d(X, n)$ can a priori have much more complicated fundamental groups.

When $d = 1$, $X = \mathbb{R}^2$, $\pi_1(B(\mathbb{R}^2, n))$ is Artin's braid group

$B(S^1, 2)$ is the open mobius band and so $B(S^1, 2) \simeq S^1$.

$B(S^1 \vee S^1, 2)$ consists of configurations of two points, and is the colimit of three subspaces: 2 points on the first leaf, 2 points on the second leaf and 1 point on each leaf. (related to Graph configuration spaces and AGVs)

$B(S^1 \vee S^1, 2)$
Proposition:

\[ B(\vee^3 S^1, 2) \]

\[ B(\vee^4 S^1, 2) \]

B(\vee^k S^1, 2) has the homotopy type of \[ \bigvee \left( \frac{3}{2} k(k-1)+1 \right) S^1 \]

Remark: For a graph \( \Gamma \), homdim\( B(\Gamma, n) \leq \#\Gamma_0 \), where \( \Gamma_0 \) is the number of essential vertices. So graphs with one essential vertex have their configuration spaces of the homotopy type of a bouquet of circles.
Main Example

Consider

$$B^{n-1}(X, n) = \text{SP}^n X - X$$

where \(X\) is the thin diagonal. We will discuss \(B^{n-1}(\mathbb{R}^k, n)\).

Start with \(\Delta^{n-1}(\mathbb{R}^k, n) = (\mathbb{R}^k)^n - \mathbb{R}^k \cong S^{k(n-1)-1}\).

This is the unit sphere in the orthogonal complement of diagonal

$$\{(v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \mid \sum v_i = 0\}$$

There is an action by \(G_n\)

$$Q_{n,k} := S^{k(n-1)-1}/G_n$$

**Theorem (Armstrong)**

Let \(G\) be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space \(X\), and let \(H\) be the normal subgroup of \(G\) generated by those elements which have fixed points, then the fundamental group of the orbit space \(X/G\) is isomorphic to the factor group \(G/H\).
Apply it to the action of $G = \mathfrak{S}_n$ on

$$\{(v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \mid \sum v_i = 0\}$$

The fixed points of the permutation action are of the form $(v_1, \ldots, v_n)$ with $v_i = v_j$ for some $i < j$, which means that all transpositions are in $H$ and hence $G = H$.

$Q_{n,k} \cong B^{n-1}(\mathbb{R}^k, n)$ is simply-connected, for $n \geq 3$.

**Theorem (K, Saihi)**

Let $X$ be a connected simplicial complex which is not reduced to a point, $n \geq 2$, $d \geq 2$. then there is an isomorphism

$$\pi_1(B^d(X, n)) \cong H_1(X; \mathbb{Z})$$

**Motto:**

*a single collision ($d \geq 2$) is enough to abelianize fundamental group*
This is based on the following local to global principle, and on eliminating locally the braiding

**Localization Principle:**
Let $X$ be a Hausdorff topological space and $Y$ be a closed subset of $X$. If for every point $y \in Y$, and every neighborhood $U \subseteq X$ of $y$, there is an open $V \subseteq U$ containing $y$ such that the pair $(V, V \setminus Y)$ is $k$-connected, $k \geq 0$, then the pair $(X, X \setminus Y)$ is $k$-connected.

We recall a pair $(X, A)$ is $k$-connected means every map from the closed cube

$$(I^r, \partial I^r) \longrightarrow (X, A) \quad , \quad 1 \leq r \leq k$$

is homotopic rel the boundary to a map $I^r \longrightarrow A$
This same principle can be used to prove the following more general statement for manifolds

**Theorem (K, Saihi)**

*If X is a simply connected manifold of dimension \( m \geq 1 \), \( n \geq 2 \) and \( 1 \leq d \leq n \), then*

\[
\pi_i(B^d(X, n)) \cong \tilde{H}_i(X; \mathbb{Z}) \quad \text{for} \quad 0 \leq i \leq 2d - 2
\]

Proof relies on three steps:

- By the localization principle, suffices to show that for every \( x \in X \), \( U \) neighborhood of \( x \), there exists sub-neighborhood \( V \subset U \) with \( B^d(V, k) \) 2\( d \) - 2-connected.

- Compute the connectivity of \( B^d(\mathbb{R}^m, n) \) for various \( d, m, n \).
Use theorem of (Dold-Puppe, K.)

**Theorem** (Dold-Puppe, K). For $r$-connected $X$, $r \geq 1$,

$$\pi_i(\text{SP}^nX) \cong \tilde{H}_i(X; \mathbb{Z}), \quad 0 \leq i \leq r + 2n - 1$$

**Proposition:** $B^d(\mathbb{R}^m, n)$ is $2d - 2$-connected

The proof is homotopy theoretic and uses *scanning maps*.

It is based on the following few steps:
- $B^d(\mathbb{R}^m, d + 1)$ is $2d - 2$-connected.

- We have inclusion maps

\[ B^d(\mathbb{R}^m, n) \rightarrow B^d(\mathbb{R}^m, n + 1) \]

sending $[x_1, \ldots, x_n] \mapsto [x_1, \ldots, x_n, \sum |x_i| + 1]$.

- These maps induce homology embeddings ($\exists$ transfer maps).

- There is a scanning map from the direct limit

\[ s : B^d(\mathbb{R}^m, \infty) \rightarrow \Omega_*^m \text{SP}^d(S^m) \]

(to a component of the loop space).

- $\Omega_*^m \text{SP}^d(S^m)$ is $2d - 2$-connected.

- The scanning map is a homology isomorphism.
For the loop space
\[ \pi_i(\Omega^m_* \text{SP}^d S^m) \cong \pi_{i+m}(\text{SP}^d(S^m)) \cong H_{i+m}(S^m) \]
for \( i + m \leq (m - 1) + 2d + 1 = m + 2d - 2 \).
“Dold-Puppe” give precisely what is needed!

Recall the space on the far left is
\[ B^d(\mathbb{R}^m, d + 1) = (\mathbb{R}^m)^{d+1} - \mathbb{R}^m \]

To find its connectivity we need the following main computation:
**Theorem:** (K, Karoui)

Let $S$ be the unit sphere in

$$\{(v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \mid \sum v_i = 0\}$$

and let $Q_{n,k}$ be its quotient under the $\mathfrak{S}_n$-action. Then

$$\Sigma^{k+1}Q_{n,k} \simeq \overline{SP}^n(S^k)$$

where $\Sigma$ means suspension and $\overline{SP}^n(Y)$ means the "symmetric smash" $Y^{\wedge(n)}/\mathfrak{S}_n$.

The case $n = 2$ of the theorem recovers an old observation.

Here $\Sigma^{k+1}Q_{2,k} \simeq \overline{SP}^2(S^k)$
Not hard to see

\[ \mathbb{S}P^2(\Sigma X) \simeq \Sigma \text{sym}^*{2} X \]

where \text{sym}^*{2} is the symmetric join.

On other hand \( Q_{2,k} \) is the quotient of the unit sphere \( S^{k-1} \) in \( \mathcal{W} = \{ (v, -v) \in (\mathbb{R}^k)^2 \} \). The generator of \( \mathbb{Z}_2 \) acts on \( \mathcal{W} \) by permuting \( v \) and \( -v \) and hence is multiplication by \( -1 \) on that sphere. This is the antipodal action.

**Corollary** (James, Thomas, Toda, Whitehead)

\[ \text{sym}^*{2}(S^k) = \Sigma^{k+1}\mathbb{R}P^k \]

where \text{sym}^*{2} is the symmetric join.
Ordered configurations

We say a space is $r$-admissible if for any $x \in X$ and any neighborhood $U$ of $x$, there is an open $V \subset U$ of $x$ such that $V - \{x\}$ is $r$-connected.

**Example:** A manifold of dimension $m$ is $m - 2$ admissible.

**Theorem:** (K, Saihi)

Let $X$ be a locally finite simplicial complex which is $r$-admissible, $r \geq 0$, $d \geq 1$. Then

$$\pi_i(\Delta^d(X, n)) \cong \pi_i(X)^n \quad \text{for} \quad i \leq rd + 2d - 2$$
This relies on a theorem of Steve Smale relating the connectivity of fibers and base to that of total space (refinement of Begle and Vietoris).

**Theorem** (Smale)

*Let* $X$ *and* $Y$ *be connected, locally compact, separable metric spaces, and let* $X$ *be locally contractible. Let* $f$ *be a mapping of* $X$ *into* $Y$ *for which* $f^{-1}$ *carries compact sets into compact sets. If, for each* $y \in Y$, $f^{-1}(y)$ *is locally contractible and* $r$-connected, *then the induced homomorphism* $\pi_k(X) \to \pi_k(Y)$ *is an isomorphism onto for* $0 \leq k \leq r$, *and is onto for* $k = r + 1$.

**Caveat:** Need compactify complements of diagonal subspaces before being able to use theorem (!)
Stable Splittings

**Theorem** (Labassi, JHRS 2013)
Let $X$ be connected of the homotopy type of a CW-complex, and let $n < 2d$. After a single suspension we have the splitting

$$
\Sigma \Delta_d(X, n) \simeq \bigvee_{m=1}^{n-d} \left( \bigvee \Sigma X^m \right) \vee \bigvee \Sigma X^{n-d+1}
$$

where $X^m$ is the smash product of $m$ copies of $X$.

**Remark 1**: Splitting no longer valid when $n \leq 2d$ (!)
The homology of $\Delta_d(X, n)$ can have torsion even if $H_*(X)$ doesn’t.

**Remark 2**: Theorem of Labassi has been extended to more general diagonal arrangements by Iriye and Kishimoto.
Euler Characteristics

Suppose $X$ is of the homotopy type of a finite simplicial complex having Euler characteristic $\chi := \chi(X)$, and let $2 \leq d \leq n$. Then

**Theorem** (K, Taamallah):

$$
\chi(B_d(X, n)) = \left(\frac{\chi + n - 1}{\chi - 1}\right)
- \sum_{\sum_{i=1}^{d-1} i\alpha_i = n} \frac{1}{\alpha_1! \cdots \alpha_{d-1}!} \chi(\chi - 1) \cdots (\chi - \sum \alpha_i + 1)
$$

where sums are taken over all sequences of non-negative integers $(\alpha_1, \ldots, \alpha_{d-1})$ such that $\sum i\alpha_i = n$. 
This is the SPECIAL case of a general theorem which gives conditions on when a filtration in topology

\[ F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \]

is additive for \( \chi \); i.e. when

\[ \chi(F_n) = \sum_{i=1}^{n} \chi(F_i - F_{i-1}) \]

This can be applied to computing \( \chi \) for \( F_n = \text{SP}^n(X), B_d(X, n), \text{Sub}_n(X) \) (the finite subsets spaces), etc.

For example:

\[ \chi(\text{SP}^3X) = \chi(\text{SP}^3_3(X)) + \chi(\text{SP}^3_{2,1}(X)) + \chi(\text{SP}^3_{1,1,1}(X)) \]

where \( \text{SP}^3_{2,1}(X) \) consists of configurations \([x, x, y], x \neq y\), and \( \text{SP}^3_{1,1,1}(X) = B(X, 3) \).
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