String Topology of Classifying Spaces

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Manifolds, $K$-theory, and Related Topics
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Joint work with Richard Hepworth
$\mathbb{Z}/2$ coefficients
Part 1: HCFTs
String topology studies algebraic structures on $H_*(LX)$. ($LX = \text{map}(S^1, X)$)

**Theorem (Chas and Sullivan 1999)**

*Suppose $M$ is a closed $d$-manifold. Then $H_{*-d}(LM)$ is a BV algebra.*

**Theorem (Godin 2007)**

*Suppose $M$ is a closed $d$-manifold. Then $H_*(LM)$ is the value on $S^1$ in a degree $d$ Homological Conformal Field Theory (HCFT).*
Definition of HCFT

Rough Definition

An HCFT $\Phi$ of degree $d$ is an assignment

$$(1\text{-manifold } X) \mapsto (\text{graded vector space } \Phi_*(X))$$

$$(\text{cobordism } \Sigma: X \to Y) \mapsto (H_*(B\text{Diff}(\Sigma)) \otimes \Phi_*(X) \to \Phi_* + d\chi(\Sigma, X)(Y))$$

compatible with disjoint unions and composition of cobordisms.

Here

$$\text{Diff}(\Sigma) = \left\{ \begin{array}{l}
\text{orientation-preserving self-diffeos of } \Sigma \\
\text{fixing } X \text{ and } Y \text{ pointwise}
\end{array} \right\}. $$
Two perspectives on HCFTs

\[
\left( \text{cobordism} \, \Sigma : X \to Y \right) \mapsto \left( H_*(BDiff(\Sigma)) \otimes \Phi_*(X) \to \Phi_{*+\text{shift}}(Y) \right)
\]

1. An HCFT $\Phi$ is an algebraic structure on $\Phi_*(S^1)$.

- A closed 1-manifold $X$ is isomorphic to $\bigsqcup^p S^1$ for some $p$, and then $\Phi_*(X) \cong \Phi_*(S^1)^{\otimes p}$.
- Each $z \in H_*(BDiff(\Sigma))$ for $\Sigma : \bigsqcup^p S^1 \to \bigsqcup^q S^1$ gives an operation
  \[
  \Phi_*(S^1)^{\otimes p} \to \Phi_*(S^1)^{\otimes q}.
  \]

  Eg. the generators of $H_0BDiff$ \(\begin{tikzpicture} \draw[dashed] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle; \end{tikzpicture}\) and $H_0BDiff$ \(\begin{tikzpicture} \draw[dashed] (0,0) -- (0,1) -- (1,1) -- (1,0) -- cycle; \end{tikzpicture}\) make $\Phi_*(S^1)$ into an algebra and a coalgebra.

- This algebraic structure sheds light on $\Phi_*(S^1)$, for example $H_*(LM)$. 

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Two perspectives on HCFTs (continued)

\[
\left( \text{cobordism} \right)_{\Sigma : X \to Y} \mapsto (H_*(BDiff(\Sigma)) \otimes \Phi_*(X) \to \Phi_{*+\text{shift}}(Y))
\]

2. An HCFT is a ‘representation’ of \( H_*(BDiff(\Sigma)) \)’s.

- \( BDiff(\Sigma) \)’s are very interesting spaces: they classify bundles with fibre \( \Sigma \), and under mild conditions on \( \Sigma \),
  - \( \text{Diff}(\Sigma) \simeq \text{mapping class group of } \Sigma \)
  - \( BDiff(\Sigma) \simeq \text{moduli space of Riemann surfaces modelled on } \Sigma \).
- HCFTs give a tool for studying the homology of these spaces.

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String Topology of $BG$

**Theorem (Godin 2007)**

Suppose $M$ is a closed manifold. Then $H_\ast(LM)$ is the value on $S^1$ in an HCFT of degree $\dim(M)$.

**Theorem (Chataur and Menichi 2007)**

Suppose $G$ is a compact Lie group. Then $H_\ast(LBG)$ is the value on $S^1$ in an HCFT of degree $-\dim(G)$.

As stated, the results seem exactly analogous...
...but in fact they differ significantly in details.

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<th>Godin</th>
<th>C–M</th>
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<tbody>
<tr>
<td>type of cobordisms</td>
<td>open–closed</td>
<td>closed only</td>
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Closed and open–closed cobordisms

Closed: incoming and outgoing boundaries consist of circles ("closed strings")

\[ \Phi_*(S^1) \]

Open–closed: also allow intervals ("open strings") as incoming and outgoing boundaries

\[ \Phi_*(S^1) \text{ and } \Phi_*(I) \]
Comparison (continued)

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<tr>
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<tbody>
<tr>
<td><strong>unit for $\Phi_*(S^1)$</strong></td>
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<td>✗</td>
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<tr>
<td><strong>counit for $\Phi_*(S^1)$</strong></td>
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<tr>
<td><strong>unit for $\Phi_*(I)$</strong></td>
<td>✓</td>
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<tr>
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Here is Chataur and Menichi’s result again:

**Theorem (Chataur and Menichi 2007)**

Suppose $G$ is a compact Lie group. Then $H_\ast(LBG)$ is the value on $S^1$ in an HCFT of degree $-\dim(G)$.

Our first theorem is similar:

**Theorem (Hepworth and L)**

Suppose $G$ is a compact Lie group. Then $H_\ast(LBG)$ is the value on $S^1$ in an HCFT of degree $-\dim(G)$.
The HCFT we construct extends Chataur and Menichi’s HCFT.

<table>
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<th>H–L</th>
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Q: Is it possible to turn any of the ✗’s into ✓’s the H–L-column?  
A: No. “Closest possible analogue to Godin’s theory.”

An extension of C–M to an open–closed theory (without units or counits) has also been constructed by Guldberg. Addition of counits is the harder part – requires new techniques!
Part 2: Beyond HCFTs

Idea: instead of surfaces and diffeomorphisms, work with homotopy graphs and homotopy equivalences.
H-graphs and h-graph cobordisms

Work towards the definition a novel kind of field theory – “A Homological H-Graph Field Theory”

Definition

An *h-graph* $X$ is a space homotopy equivalent to a finite graph.

Examples: $S^1$, $S^1 \vee S^1$, $I$, connected compact surfaces with non-empty boundary.

Definition

An *h-graph cobordism* $S: X \rightarrow Y$ is a diagram $X \leftrightarrow S \leftrightarrow Y$ of h-graphs satisfying certain conditions.

Example: An ordinary cobordism $\Sigma: X \rightarrow Y$ between 1-manifolds with the property that all components of $\Sigma$ meet $X$. 
For an h-graph cobordism \( S: X \to Y \), denote

\[
h\text{Aut}(S) = \left\{ \text{self-homotopy equivalences of } S \middle| \text{fixing } X \text{ and } Y \text{ pointwise} \right\}.
\]

**Rough Definition**

A *Homological H-Graph Field Theory (HHGFT)* \( \Phi \) of degree \( d \) is an assignment

\[
(h\text{-graph } X) \mapsto (\text{graded vector space } \Phi_*(X))
\]

\[
(h\text{-graph cob } S: X \to Y) \mapsto (H_*(Bh\text{Aut}(S)) \otimes \Phi_*(X) \to \Phi_{*+d}(X, Y))
\]

compatible with disjoint unions and composition of cobordisms.
An HHGFT restricts to an HCFT:
- $S^1$ and $I$ are h-graphs
- An ordinary cobordism $\Sigma: X \to Y$ is an h-graph cobordisms (as long as $X$ meets every component of $\Sigma$)
- We have a natural map $\text{Diff}(\Sigma) \to \text{hAut}(\Sigma)$

**Theorem (Hepworth and L)**

*The HCFT from our first theorem extends to an HHGFT $\Phi^G$.*

On h-graphs, the theory is given by $\Phi^G_*(X) = H_*\text{map}(X, BG)$. 
Some consequences and benefits

- New cobordisms $\leadsto$ new operations
  
  Example:

  \[ \phi: \mathbb{S}^1 \to I \leadsto \text{new operation } \Phi_*(\mathbb{S}^1) \to \Phi_*(I) \]

- New factorizations of existing cobordisms
  
  Example:

  \[ \begin{array}{c}
    \includegraphics[width=1.5cm]{cobordism1} \\
    \quad = \quad \includegraphics[width=1.5cm]{cobordism2} \circ \includegraphics[width=1.5cm]{cobordism3}
  \end{array} \]

- Operations parametrized by homologies of automorphism groups of free groups (with boundaries)
The automorphism group of free group on $n$ generators with $k$ boundary circles and $s$ boundary points is

$$A^s_{n,k} = \pi_0 \text{hAut}(\Gamma^s_{n,k}; \partial)$$

where

$$\Gamma^s_{n,k} = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}$$

and $\partial = \text{---}$. 

- $A^1_{n,0} \approx \text{Aut}(F_n)$
- $A^2_{n,0} \approx F_n \rtimes \text{Aut}(F_n) = \text{Hol}(F_n)$
- $A^1_{0,k}$ is a central extension by $\mathbb{Z}^k$ of the pure symmetric automorphism group of $F_k$. 

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Operations parametrized by $H_*(BA_{n,k}^s)$

$$A_{n,k}^s = \pi_0 h\text{Aut}(\Gamma_{n,k}^s; \partial); \quad \Gamma_{n,k}^s = \quad \partial = \quad$$

- If $\partial \neq \emptyset$, we can turn $\Gamma_{n,k}^s$ into an h-graph cobordism by dividing $\partial$ into incoming and outgoing parts.

- The HHGFT now gives operations parametrized by $H_*(Bh\text{Aut}(\Gamma_{n,k}^s; \partial))$.

- The components of $h\text{Aut}(\Gamma_{n,k}^s; \partial)$ are contractible.

- Therefore $h\text{Aut}(\Gamma_{n,k}^s; \partial) \simeq A_{n,k}^s$, and we get operations parametrized by $H_*(BA_{n,k}^s)$. 

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Let $S_n : \text{pt} \to \text{pt}$ be the h-graph cobordism $\bullet \to \bullet$. 

$\Sigma_n$ injects into $\text{hAut}(S_n)$ as permutations of the $n$ strings. Let $\varphi^G_n$ be the composite

$$\varphi^G_n : H_* B\Sigma_n \otimes H_* BG \to H_* Bh\text{Aut}(S_n) \otimes H_* BG \xrightarrow{\Phi^G(S_n)} H_* + \text{shift} BG.$$ 

**Theorem (Hepworth and L)**

The map $\varphi^Z_2 : H_* B\Sigma_2 \otimes H_* B(\mathbb{Z}/2) \to H_* B(\mathbb{Z}/2)$ is given by $a \otimes b \mapsto \begin{cases} a \cdot b & \text{if the degree of } a \text{ is positive} \\ 0 & \text{if the degree of } a \text{ is 0} \end{cases}$

Recall that $H_* B\Sigma_2$ is a ring: $H_* B\Sigma_2 \simeq \Gamma(x)$, $|x| = 1$. Canonical iso $\mathbb{Z}/2 \simeq \Sigma_2$ makes $H_* B(\mathbb{Z}/2)$ into a $H_* B\Sigma_2$-module.
Theorem (Hepworth and L)

The map $\varphi^\mathbb{Z}/2: H_\ast B\Sigma_2 \otimes H_\ast B(\mathbb{Z}/2) \rightarrow H_\ast B(\mathbb{Z}/2)$

is given by $a \otimes b \mapsto \begin{cases} a \cdot b & \text{if the degree of } a \text{ is positive} \\ 0 & \text{if the degree of } a \text{ is 0} \end{cases}$

- Gives an infinite family of non-trivial higher string topology operations – one for each non-zero $a \in H_\ast B\Sigma_2$, $|a| > 0$.
- For $|a| > 1$, these operations cannot arise from any HCFT operation.
Calculations (continued)

L: Further calculations of $\varphi^G_n : H_* B\Sigma_n \otimes H_* BG \to H_{*+\text{shift}} BG$:

- for $G = (\mathbb{Z}/2)^k$, $D_{4k+2}$ and all $n$
- for $G = \mathbb{T}^k$, $SU(2)$ and small $n$

Get interesting operations for all these $G$. For example:

**Theorem (L)**

The map $\varphi^{SU(2)}_2 : H_* B\Sigma_2 \otimes H_* BSU(2) \to H_{*+3} BSU(2)$ is given by $a_k \otimes b \mapsto a_{k+3} \cdot b$

Here $a_k$ denotes the non-trivial class in $H_k B\Sigma_2 \approx \mathbb{Z}/2$, $k \geq 0$. $H_* BSU(2)$ is made into a $H_* B\Sigma_2$-module using the map

$$\Sigma_2 \times SU(2) \to SU(2), \quad (\sigma, A) \mapsto \begin{cases} A & \text{if } \sigma = \text{id} \\ -A & \text{if } \sigma = (12) \end{cases}$$
The calculations give lots of examples of non-trivial string topology operations associated with $S_n : \text{pt} \to \text{pt}$.

More generally, get lots of non-trivial operations for composites $S_{n_1} \circ \cdots \circ S_{n_k} : \text{pt} \to \text{pt}$.

**Corollary**

Get non-trivial classes in

$$H_q B\text{hAut}(S_{n_1} \circ \cdots \circ S_{n_k}) \approx H_q B\text{Hol}(F_N)$$

for various $q$'s. Here $N = \sum_{i=1}^{k} (n_i - 1)$.

**Corollary**

$$H_{q-1}(B\text{Aut}(F_N); \tilde{F}_2^N) \neq 0 \text{ for these } q.$$ 

For $G$ abelian, the elements in $H_q B\text{Hol}(F_N)$ survive to $H_q B\text{Aff}_N(Z)$, where $\text{Aff}_N(Z) = \text{Hol}(Z^N) = Z^N \rtimes GL_N(Z)$. 
Other examples of HHGFTs?

Are there other examples of HHGFTs?

**Conjecture**

*Godin’s HCFT in string topology of manifolds extends to an HHGFT.*