On the map of Bökstedt–Madsen from the cobordism category to $A$-theory

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The Bökstedt–Madsen map

If \((W; M, N)\) is a \(d\)-dimensional cobordism between closed manifolds, then

\[
BO(d) \cup_M W \leftrightarrow BO(d)
\]

is a (homotopy finite) retractive space over \(BO(d)\).

**Definition (incorrect)**

The Bökstedt–Madsen map

\[
\tau : \Omega BC_d \rightarrow A(BO(d))
\]

is induced by

\[
N_n C_d \rightarrow S_n R^f(BO(d))
\]

which sends a filtered cobordism to a filtered retractive space.
Characteristic classes in $A$-theory

A closed $d$-manifold $M$ gives rise to a loop in $BC_d$. This defines

$$i : BDiff(M) \to \Omega BC_d.$$ 

Thus the Bökstedt–Madsen map gives rise to a characteristic class

$$\tau(p) : B \to A(BO(d))$$

for bundles $p : E \to B$ of closed $d$-manifolds.

**Theorem 1 [Raptis–S.]**

This characteristic class agrees with the $A$-theory Euler class $\chi(p)$ by [Dwyer–Weiss–Williams]: For each closed smooth $d$-manifold $M$, the following triangle commutes in the homotopy category:

$$
\begin{array}{ccc}
BDiff(M) & \xrightarrow{i} & \Omega BC_d \\
\downarrow{\chi(p_{univ})} & & \downarrow{\tau} \\
A(BO(d)) & & \\
\end{array}
$$
**The index theorem**

**Index Theorem [Dwyer–Weiss–Williams]**

If $p: E \to B$ is a smooth fiber bundle of compact manifolds, then $\chi(p)$ agrees with the composite

\[
B \xrightarrow{\text{transfer}} Q(E_+) \xrightarrow{T_{\text{fib}}(E)} Q(BO(d)_+) \xrightarrow{\eta} A(BO(d)).
\]

This suggests:

**Theorem 2 [Raptis–S.]**

The Bökstedt–Madsen map factors as

\[
\Omega BC_d \xrightarrow{\text{transfer}} Q(BO(d)_+) \xrightarrow{\eta} A(BO(d)_+).
\]

In their paper, [Bökstedt–Madsen] expected both Theorems 1 and 2 to hold.
Continuity

Recall the naive construction

\[ N_nC_d \to S_nR^f(BO(d)). \]

Problem: The cobordism category is a topological category.
Solution: Make the right-hand side topological (actually, simplicial).

Definition [Williams]

Let \( p: E \to B \) be a fibration. A retractive space over \( p \) is a retractive space \( X \leftrightarrow E \) over \( E \), such that \( X \) is a fibration over \( B \).

We get a Waldhausen category \( R^f(p) \) of (homotopy finite) retractive spaces over \( p \) and we let

\[ A(p) = \Omega|wS \bullet R^f(p)|. \]
Bivariant $A$-theory

1. For $f : B' \to B$, we get

$$f^* : A \left( \frac{f^* E}{B'} \right) \to A \left( \frac{E}{B} \right).$$

2. For a map $g : E \to E'$ of fibrations over $B$, we get

$$g_* : A \left( \frac{E}{B} \right) \to A \left( \frac{E'}{B} \right).$$

3. Product pairing

$$\circ \circ : A \left( \frac{E}{B} \right) \wedge A \left( \frac{X}{E} \right) \to A \left( \frac{X}{B} \right).$$

Push–forward and pull–back are functorial, homotopy invariant, and commute with each other. (Compare [Fulton–MacPherson].)
The Bökstedt–Madsen map

We replace the naive construction

\[ N_n C_d \rightarrow S_n R^f(BO(d)) \]

by

\[ \text{sing}_* N_n C_d \rightarrow S_n R^f \left( \frac{BO(d) \times \Delta^\bullet}{\Delta^\bullet} \right). \]

This leads to

\[ \Omega \mid \text{sing}_* N_n C_d \mid \rightarrow \Omega \mid wS_\bullet R^f \left( \frac{BO(d) \times \Delta^\bullet}{\Delta^\bullet} \right) \mid \]

\[ BDiff(M) \xrightarrow{i} \Omega BC_d \xrightarrow{\tau} A(BO(d)). \]
Coassembly [Weiss–Williams]

A coassembly map is a map

\[ c: A \left( \frac{B \times F}{B} \right) \rightarrow A'(B, F) \]

such that

1. \( c \) is natural in \( B \),
2. \( c \) is a weak equivalence for \( B = \{\ast\} \), and
3. \( A' \) is cohomological in \( B \).

An example is the scanning construction

\[ A \left( \frac{B \times F}{B} \right) \rightarrow \text{map} \left( | \text{sing} \bullet B |, \left| A \left( \frac{\Delta^\bullet \times F}{\Delta^\bullet} \right) \right| \right), \]

\[ x \mapsto [(f: \Delta^n \rightarrow B) \mapsto f^* x] \]

Any two coassembly maps are canonically weakly equivalent.
Comparison with the Euler class

**Observation**

On $BDiff(M)$, the Bökstedt–Madsen map is the scanning construction of the class

$$[(BO(d) \times BDiff(M)) \sqcup E_{univ} \leftrightarrow BO(d) \times BDiff(M)]$$

$$\in A \left( \begin{array}{c} BO(d) \times BDiff(M) \\ BDiff(M) \end{array} \right).$$

**Lemma**

The $A$-theoretic Euler class agrees with a (different) coassembly on the same class.

Theorem 1 follows by uniqueness of coassembly.
Cobordism category with boundaries

Let $C_{d,\partial}$ be the cobordism category with boundaries:
- objects are compact $(d-1)$-manifolds possibly with boundary,
- morphisms are cobordisms of these.

**Theorem [Genauer]**

The transfer map is an equivalence

$$\Omega BC_{d,\partial} \xrightarrow{\simeq} Q(BO(d)_+)$$
Factorization over $Q(BO(d)_+)$

The Bökstedt–Madsen map extends to

$$
\begin{array}{cccc}
\Omega BC_d & \xrightarrow{\tau} & A(BO(d)) \\
\downarrow & & \downarrow \\
\Omega BC_{d,\partial} & \xrightarrow{\tau_{\partial}} & \\
\end{array}
$$

We are left to show that the following diagram commutes:

$$
\begin{array}{cccc}
\Omega BC_{d,\partial} & \xrightarrow{\text{transfer}} & Q(BO(d)_+) \\
\uparrow^{\tau_{\partial}} & & \uparrow^{\eta} \\
A(BO(d)) & \xleftarrow{\tau} & \\
\end{array}
$$

A reduction to the case $d = 0$ leads to the case of configuration spaces and Segal’s model for the stable homotopy $Q(BO(d)_+)$. 