CONFIGURATION SPACE INTEGRALS: BRIDGING PHYSICS, GEOMETRY, AND TOPOLOGY OF KNOTS AND LINKS

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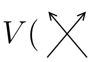
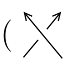
Abstract. This note contains a summary of the significance and influence of configurations space integrals (or Bott-Taubes integrals) in knot theory and the theory of embeddings more generally. It was written as an introduction to the collection of papers written by Raoul Bott on this subject, to be included in the fifth volume of his collected works.

Early 1990s witnessed an emergence of new techniques and points of view in the study of spaces of knots and spaces of embeddings more generally. One of the most exciting developments was the introduction of *finite type* or *Vassiliev* knot invariants [Vas90]. To explain, any knot invariant V can be extended to singular knots with n transverse double points via the repeated use of the *Vassiliev skein relation*:

$V($  $) = V($  $) - V($  $).$

Then V is a *type n invariant* if it vanishes identically on knots with $n+1$ double points (this definition first appeared in [BL93]). Thus finite type invariants are, in some sense, those invariants that behave like polynomials when extended to singular knots. The excitement surrounding these invariants was the (yet unresolved at the time of writing) conjecture that finite type invariants separate knots. This conjecture arose from various interesting properties of finite type invariants, including their many connections to physics and combinatorics – finite type invariants were quickly connected to certain trivalent diagrams that were reminiscent of Feynman diagrams familiar from physics (standard introductory literature on finite type invariants that gives more detail on this is [BN95, CDM12]). In 1993, Kontsevich [Kon94] proved a remarkable theorem that the space of finite type invariants is in fact isomorphic to the dual of the space of these diagrams modulo some relations. This is now known as the Fundamental Theorem of Finite Type Invariants. The isomorphism is realized by the famous *Kontsevich Integral*, a beautiful but difficult construction that was as ingenious as it was mysterious at the time of its inception.

Raoul Bott was intrigued by Kontsevich's theorem and wanted to understand it from a more classical point of view. Kontsevich's construction indicated that the cohomology of configuration spaces should play a role and some hints that this might indeed be the case had already appeared in the work of Guadagnini, Martellini, and Mintchev [GMM89], as well as Bar-Natan [BN], whose approach was inspired by Chern-Simons Theory. Bott started talking to Cliff Taubes (who occupied the office next to his) about this, and the result of their conversations was their 1994 seminal paper "On the self-linking of knots" [BT94] which paved a way for two decades of active research in applications of configuration space integrals.

The guiding idea of [BT94] was that the familiar linking number of two-component links, given by the Gauss integral that essentially counts the number of times one link strand crosses another in a projection, with a sign, should be adaptable to give an invariant (or a family of invariants) of knots. Mimicking the Gauss integral, the first natural idea is to consider two points moving on the knot, keep track of the direction between them, and then use this direction map to pull back the canonical volume form from S^2 and integrate it over $\mathrm{Conf}(2, \mathbb{R})$, the configuration space of two points in $\mathbb{R}$. If all goes well, the resulting pushforward form, which is a zero-form on the space of knots, would be an invariant of knots just like the linking number of two-strand links.

It turns out that this does not quite work out because one does not obtain a closed form in this way. However, Bott and Taubes decided to study the next potentially interesting case, that of four points on the knot with two maps to S^2 keeping track of the directions between two pairs of points.

Namely, let $\mathcal{K}$ be the space of knots in $\mathbb{R}^3$ (embeddings of a circle in $\mathbb{R}^3$) and let $\mathrm{Conf}[k,\mathbb{R}]$ be the *Fulton-MacPherson compactification* [AS94, FM94] of $\mathrm{Conf}(k,\mathbb{R})$ ($C_k(\mathbb{R})$ in the notation of [BT94]), the configuration space of k points in $\mathbb{R}$. The reason one needs to work with the compactifications is to ensure that the integrals converge. These compactifications are manifold with corners, and one of the main contributions of Bott and Taubes from [BT94] was to study the stratification and the parametrization of their boundary. Then we have a diagram

$$\begin{array}{ccc} \mathrm{Conf}[4,\mathbb{R}]\times\mathcal{K} & \xrightarrow{\Phi=\Phi_{13}\times\Phi_{24}} & S^2\times S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{K} & & \end{array}$$

where Φ_{ij} is the compositions of the evaluation map that evaluates a knot on the ith and the jth point (out of four ordered points) on $\mathbb{R}$, and then takes their normalized difference. Now let sym_{S^2} be the unit volume form on S^2 and let $\alpha=\Phi^*(\mathrm{sym}^2_{S^2})$. Since α and $\mathrm{Conf}[4,\mathbb{R}]$, the fiber of π, are both 4-dimensional, integration along the fiber, or pushforward, of π thus yields a zero-form $\pi_*\alpha$ on the space of knots $\mathcal{K}$.

The question now is if this form is closed, i.e. if it is an element of $\mathrm{H}^0(\mathcal{K})$, a knot invariant. By Stokes' Theorem, this question reduces to checking whether the restriction of the pushforward to the codimension one boundary of $\mathrm{Conf}[4,\mathbb{R}]$ vanishes. This boundary has various components corresponding to how points collide in the compactification. Bott and Taubes argued that, for many faces, the integral indeed vanishes, but the faces determined by two points colliding at a time (so-called *principal* faces), were still a problem.

The ingenious solution was to introduce another space that had precisely the same problematic boundary. Then the difference of integrals over $\mathrm{Conf}[4,\mathbb{R}]$ and this new space should then indeed be a closed form. It turns out that thinking geometrically about what kind of boundary this space should have leads naturally to its definition. The geometry suggests that we want a space of four configuration points in $\mathbb{R}^3$, three of which are constrained to lie on a knot, and we want to keep track of three directions between the points on the knot and the one off the knot. Then the collision of the "free" point with the points on the knot produces precisely the three principal faces from the initial setup.

To make this precise, Bott and Taubes define the pullback space

(1)
$$\begin{array}{ccc} \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \longrightarrow & \mathrm{Conf}[4,\mathbb{R}] \\ \downarrow & & \downarrow{\scriptstyle \mathsf{proj}} \\ \mathrm{Conf}[3,\mathbb{R}]\times\mathcal{K} & \xrightarrow{\mathsf{eval}} & \mathrm{Conf}[3,\mathbb{R}] \end{array}$$

where eval is the evaluation map and proj the projection onto the first there points of a configuration. ($\mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]$ is $C_{3,1}$ in the notation of [BT94].) There is now an evident map

$$\pi'\colon\ \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]\longrightarrow\mathcal{K}$$

whose fiber over a knot $K\in\mathcal{K}$ is precisely the configuration space of four points, three of which are constrained to lie on K. Bott and Taubes show that the map π' is a smooth bundle and so one can perform integration along its fiber. So let

$$\Phi=\Phi_{14}\times\Phi_{24}\times\Phi_{34}\colon\ \mathrm{Conf}[3,1;\mathcal{K}^3,\mathbb{R}^3]\longrightarrow(S^2)^3$$

be the map giving the three directions between the points on the knot and the free point. The relevant maps are thus

$$\begin{array}{ccc} \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \xrightarrow{\Phi} & (S^2)^3 \\ \downarrow \pi' & & \\ \mathcal{K} & & \end{array}$$

As before, let α' be the pullback form $\Phi^*(\mathrm{sym}^3_{S^2})$. This form can be integrated along the fiber $\mathrm{Conf}[3,1;K,\mathbb{R}^3]$ over K. Both the form and the fiber are 6-dimensional, so integration gives a zero-form $\pi'_*\alpha'$ on $\mathcal{K}$. The main result of [BT94] is

Theorem 0.1. *The zero-form*

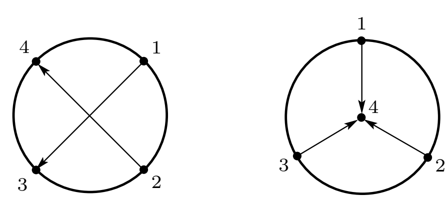
$$\pi_*\alpha - \pi'_*\alpha'$$

is closed, i.e. it is a knot invariant.

Bott and Taubes dubbed this a "self-linking" invariant because of its origins in the linking number as described above.

It turns out that same statement was proved in [BN] and [GMM89] but from very different points of view. Bott and Taubes' proof is geometrically intuitive, but it is topological and requires a deep understanding of the Fulton-MacPherson compactifications and integration along the fiber. It proceeds by a careful analysis of how the pushforward restricts to all the codimension-one faces and that all the restrictions either vanish or cancel out. For summaries of the main features of [BT94], the reader might want to consult two other papers in this collection, [Bot97] and [Bot96], both of which are based on the talks Bott gave on this subject. Some more detail can also be found in [Vol07].

A helpful feature of the Bott-Taubes construction is that the combinatorics of compactified configuration spaces is easily kept track of in terms of diagrams. The situation corresponding to integration over $\mathrm{Conf}[4,\mathbb{R}]$ can be depicted with the left diagram in the picture below and the one corresponding to $\mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]$ to the one on the right:



These diagrams, as it turns out, are precisely some of the trivalent diagrams one encounters in the theory of finite type invariants mentioned at the beginning of this note. It so happens that Dylan Thurston was a senior at Harvard the year that Bott and Taubes worked out the results of [BT94], and Bott gave Thurston the project of further investigating their self-linking invariant. Thurston not only succeeded in showing that the Bott-Taubes invariant is a finite type two invariant, but vastly generalized the construction. Namely, starting with a trivalent diagram with p points on a circle, k points off the circle, and some number of edges between them, Thurston, in analogy with $\mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]$, defines a space $\mathrm{Conf}[p,q;\mathcal{K},\mathbb{R}^3]$ and a map to the product of as many sphere as there are edges. Then integrating the pullback of the product of volume forms along the canonical map $\mathrm{Conf}[p,q;\mathcal{K},\mathbb{R}^3] \to \mathcal{K}$ produces a zero-form on $\mathcal{K}$. Thurston shows that, fixing a positive integer n and taking the sum of integrals (with signs, a normalizing factor, and modulo some relations from finite type theory) over all diagrams with $p+q=2n$ gives a type n knot invariant. Furthermore, this gives all finite type invariants (by varying a functional on the space of trivalent diagrams). Thurston's result thus gives an alternative proof of the Fundamental Theorem of Finite Type Invariants, i.e. an alternative to the Kontsevich Integral.

Thurston's work appears in [Thu], but was unfortunately never published (a recounting and a slight expansion of Thurston's work appears in [Vol07]). Nevertheless, his results were very influential in the theory of finite type invariants. For example, his construction was generalized to links and homotopy links in [KMV13], were used for placing finite type invariants in the context of calculus of functors in [Vol06], and were even extended to the setting of invariants of vector fields in [KV]. Recent work [KKV] also gives an interpretation of Milnor invariants, classical objects in knot theory, via Thurston-like configuration space integrals indexed on trees.

The next generalization of the Bott-Taubes integrals was performed by Cattaneo, Cotta-Ramusino, and Longoni in [CCRL02]. The idea there was that there was no reason to restrict the attention to trivalent diagrams and they now allowed the vertices of the diagram to have valence greater than three. The rest of Thurston's setup remained the same, but the result was that now the cohomology classes produced on $\mathcal{K}$ were not just zero-dimensional. Furthermore, one could this way obtain cohomology classes of $\mathcal{K}^n$, spaces of knots in $\mathbb{R}^n$ for $n > 3$ (even though there is no knotting in these spaces, they are very interesting and have rich topology). It was then shown in[CCRL02] that, for $n > 3$, there is a cochain map

$$\mathcal{D}^{k,m} \longrightarrow \Omega^{(n-3)k+m}(\mathcal{K}^n)$$

between the bigraded complex of at least trivalent diagrams on the left (with the interesting differential given by contractions of edges) and the complex of deRham forms on $\mathcal{K}^n$ on the right. In the subsequent work [CCRL05], Cattaneo, Cotta-Ramusino, and Longoni use this to show that, given any $i > 0$ and $n > 3$, $\mathcal{K}^n$ has nontrivial cohomology in dimension greater than i. Algebraic structure on the complex of diagrams $\mathcal{D}^{k,m}$ were further examined in [Lon04].

For a survey of the original Bott-Taubes construction, Thurston's generalization, and Cattaneo, Cotta-Ramusino, and Longoni work, the reader might be interested in consulting [Vol13].

That Cattaneo was involved with configuration space integrals was no accident. He was a postdoc at Harvard when Bott approached him after one of Cattaneo's talks. Bott wanted to understand the perturbative Chern-Simons 3-manifold invariants by extending the work he and Taubes did in [BT94]. This resulted in a collaboration that produced the papers [BC98, BC99]. In these articles, Cattaneo and Bott produce invariants of manifolds M with framing f and a fixed Riemannian metric:

$$I_\Gamma(M, f) = A_\Gamma(M) + \phi(\Gamma)CS(M, f),$$

where Γ is a diagram, $A_\Gamma(M)$ is a configuration space integral in the spirit of those in [BT94], $\phi(\Gamma)$ is a real number associated to Γ, and $CS(M, f)$ is the Chern-Simons integral of the Levi-Civita connection of M with respect to f. This expression depends on the framing f, but if M is a rational homology sphere, one obtains true invariants

$$J_\Gamma(M) = A_\Gamma(M) - 4\theta(\Gamma)A_\Theta(M).$$

Here $A_\Theta(M)$ is the "self-linking" integral, namely the integral over the simplest diagram with one chord connecting two vertices on the circle. The above is the main result of [BC98], while the extension to nontrivial local coefficient system was performed in [BC99]. This investigation has its roots in the work of Witten who initiated the study of invariants of manifolds out of Chern-Simons theory, and is an alternative to Kontsevich's way of doing this from [Kon94]; note that this very much mimics the situation described above where Bott and Taubes set out to understand and give an alternative approach to Kontsevich's proof of the Fundamental Theorem of Finite Type Invariants. In fact, Bott and Cattaneo's results are very much related to finite type invariants of homology spheres which has been a productive field of investigation in the last fifteen years. Cattaneo himself has recently returned to the study of configuration space integrals in the context of Chern-Simons theory [CM10].

We have already seen how the paper "On the self-linking of knots" has generated various branches of investigation – Thurston extending from four points to more and Cattaneo, Cotta-Ramusino, and Longoni generalizing from trivalent diagrams to arbitrary valence. Yet another direction is to generalize the space of knots. The work of Sakai [Sak10] and its expansion by Sakai and Watanabe [SW12] is relevant in this direction. Namely, they consider embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$ and use configuration space integrals to produce nontrivial cohomology classes of this space with certain conditions on k and n. This work generalizes classes produced by others [CR05, Wat07] and complements work by Arone and Turchin [AT14] who show, using homotopy-theoretic methods, that the homology of the space of embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$ is given by a certain graph complex for $n \geq 2k+2$. Sakai has further used configuration space integrals to produce a cohomology class of $\mathcal{K}$ in degree one that is related to the Casson invariant [Sak08] and has given a new interpretation of the Haefliger invariant for embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$ for some k and n [Sak10]. In an interesting bridge between two different points of view on spaces of knots, Sakai has in [Sak10] also combined the configuration space integrals with Budney's action of the little discs operad on $\mathcal{K}^n$ [Bud07].

Lastly, another recent interesting development is the work of Koytcheff [Koy09] who develops a homotopy-theoretic replacement of configuration space integrals. He uses the Pontryagin-Thom construction to "push forward" forms from $\mathrm{Conf}[p, q; \mathcal{K}^n, \mathbb{R}^n]$ to $\mathcal{K}^n$. The advantage of this approach is that it works over any coefficients, unlike ordinary configuration space integration, which takes values in $\mathbb{R}$. A better understanding of how Koytcheff's construction relates to the original configuration space integrals is still needed.

Bott's work on configuration space integrals has produced a wealth of exciting mathematics. The subject is increasing in popularity and the techniques are being applied to more and more situations. There are still various open questions about these integrals – Why do they appear in a seemingly unrelated subject of the rational formality of the little n-discs operad [Kon99, LV14]? Can they be used for showing that finite type invariants separate knots and links?, etc. – and their popularity is bound to grow in the years to come.

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