Some combinatorial problems arising from manifold calculus of functors

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Plan of the talk

Outline:

1. The idea of calculus of functors
2. Manifold calculus of functors
   (a) Embedding spaces
   (b) Tverberg Conjecture
3. Space of long knots
   (a) Taylor tower for the space of long knots
   (b) Application to the homology of the space of long knots
4. Spaces of links
   (a) Generalizations of results for knots to links
   (b) Homotopy string links and subspace arrangements

1. The idea of calculus of functors

Calculus of functors is a theory that aims to "approximate" functors in algebra and topology much like the Taylor polynomials approximate ordinary smooth functions. Generally, given a functor

$$F\colon \mathcal{C} \longrightarrow \mathcal{D}$$

this theory produces a "Taylor tower" of functors and morphisms:

$$\begin{array}{c} F(-) \\ \swarrow \quad \downarrow \quad \searrow \quad \searrow \\ T_0F(-) \longleftarrow \cdots \longleftarrow T_{k-1}F(-) \longleftarrow T_kF(-) \longleftarrow \cdots \longleftarrow T_\infty F(-) \end{array}$$

- T_kF is *kth Taylor polynomial* (or *stage*) *of* F;
- $T_\infty F$ is the inverse limit of the tower, i.e the "Taylor series" of F.

Depending on F, this tower *converges*, meaning that there is an equivalence, for all $X \in \mathcal{C}$,

$$F(X) \simeq T_\infty F(X).$$

(The word "equivalence" depends on the context.)

1. The idea of calculus of functors

There are currently three versions of functor calculus:

- Homotopy calculus
- Orthogonal calculus
- Manifold calculus

Each studies different kinds of functors:

- Homotopy: $X \mapsto \Sigma^\infty X,\ \Omega^\infty\Sigma^\infty X,\ X$;
- Orthogonal: $V \mapsto BO(V),\ BU(V),\ S^V,\ \Omega^V S^V$;
- Manifold: $M \mapsto \mathrm{Map}(M,N),\ \mathrm{Imm}(M,N),\ \mathrm{Emb}(M,N)$

Analogies with the Taylor series are the most apparent in homotopy calculus, where, for example, the kth polynomial is given by

$$\left(\partial_k F(*) \wedge X^{\wedge k}\right)_{h\Sigma_k}$$

which corresponds to

$$\left(f^{(k)}(0) \cdot x^k\right) \Big/ k!$$

2. Manifold calculus

We are interested in manifold calculus:

Let Top be the category of topological spaces and let M and N be smooth manifolds.

$$\begin{aligned}\mathcal{O}(M) = &\text{ category of open subsets of } M\\ &\text{ with inclusions as morphisms.}\end{aligned}$$

Manifold calculus studies functors

$$F\colon \mathcal{O}(M)^{op} \longrightarrow \text{Top}$$

For each such functor, this theory produces a Taylor tower of functors/spaces with natural transformations/fibrations between them.

Stages $T_k F$ of the Taylor tower are generally defined as follows:

2. Manifold calculus

Let $\mathcal{O}_k(-)$ be the subcategory $\mathcal{O}(-)$ consisting of open subsets of M diffeomorphic to up to k disjoint balls.

Then, for $U \subset \mathcal{O}(M)$, the kth stage of the Taylor tower is defined as

$$T_k F(U) = \underset{V \in \mathcal{O}_k(U)}{\text{holim}} F(V).$$

This homotopy limit is in some sense trying to reconstruct $F(U)$ from information about collections of its open balls (Kan extension).

In some circumstances, the canonical map $F(-) \to T_k F(-)$ induces isomorphisms on homotopy groups in a range that grows with k so that we have an equivalence

$$F(-) \xrightarrow{\simeq} T_\infty F(-).$$

(Usually the functor is then evaluated on M, i.e. we're interested in the equivalence $F(M) \xrightarrow{\simeq} T_\infty F(M)$.)

2(a). Manifold calculus of functors and embeddings

Definition

Let M and N be smooth manifolds. An *embedding* of M in N is an injective map $f\colon M \hookrightarrow N$ whose derivative is injective and which is a homeomorphism onto its image.

When M is compact, an embedding is an injective map with injective derivative.

The set of all embeddings of M in N can be topologized so we get the *space of embeddings* $\mathrm{Emb}(M, N)$.

2(a). Manifold calculus of functors and embeddings

Example

- $\mathrm{Emb}(*, N) \cong N$
- $\mathrm{Emb}(\sqcup_p *, N) = \{(x_1, x_2, ..., x_p) \in N^p : x_i \neq x_j \text{ for } i \neq j\} = \mathrm{Conf}(p, N) =$ configuration space of p points in N
- $\mathrm{Emb}(D^m, \mathbb{R}^n) \cong \mathrm{Stiefel}_m(\mathbb{R}^n)$
- $\mathrm{Emb}(S^1, \mathbb{R}^3) =$ space of knots

As is evident even from the above examples, for many M and N, the space of embeddings of M in N is a topologically interesting (and difficult) space, so we want to know

$$\pi_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}^*(\mathrm{Emb}(M, N)).$$

Even $\pi_0(\mathrm{Emb}(M, N))$, the set of *isotopy classes of* $\mathrm{Emb}(M, N)$, is often very difficult to understand.

2(a). Manifold calculus of functors and embeddings

Embedding space $\mathrm{Emb}(M, N)$ can be thought of as a contravariant functor of open subsets of M:

$$\mathrm{Emb}(-, N)\colon \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}$$

Namely, for each inclusion

$$O_1 \hookrightarrow O_2$$

of open subsets, there is a restriction

$$\mathrm{Emb}(O_2, N) \longrightarrow \mathrm{Emb}(O_1, N).$$

This is also true for the space of *immersions* $\mathrm{Imm}(M, N)$ (smooth maps with injective derivative) and $\mathrm{Map}(M, N)$ (smooth maps).

2(a). Manifold calculus of functors and embeddings

Theorem (Goodwillie-Klein-Weiss)

Let $m = dim(M)$ and $n = dim(N)$. For $n \geq m+3$ and $U \in \mathcal{O}(M)$, the map

$$\mathrm{Emb}(U, N) \longrightarrow T_k \,\mathrm{Emb}(U, N)$$

is $(k(n-m-2)+1-m)$-connected (induces isomorphisms on homotopy groups up to that number). Since this number grows with k, there is an equivalence

$$\mathrm{Emb}(U, N) \xrightarrow{\simeq} T_\infty \,\mathrm{Emb}(U, N).$$

This result is usually used with $U = M$.

Similar convergence result is true for $\mathrm{H}_*(\mathrm{Emb}(M, N))$ and $\mathrm{H}^*(\mathrm{Emb}(M, N))$, but the condition then is $n \geq 4m$.

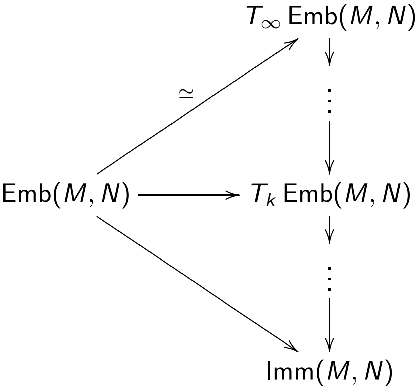
Note that when M is 1-dimensional, N has to be at least 4-dimensional in both conditions.

2(a). Manifold calculus of functors and embeddings

It is not hard to show (Smale-Hirsch) that

$$T_1 \operatorname{Emb}(M, N) \simeq \operatorname{Imm}(M, N),$$

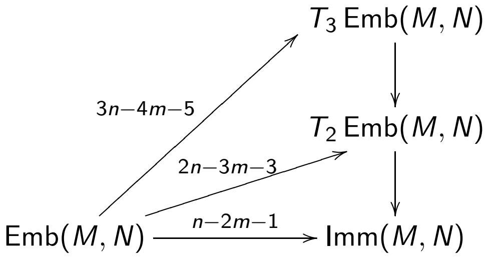
so that the Taylor tower for $\operatorname{Emb}(M, N)$ classifies obstructions for having a homotopy that starts with an immersion and ends with an embedding, in codimension ≥ 3. I.e. the Taylor tower tells us if it is possible to turn an immersion into an injective one and what the obstructions are for doing this.



2(a). Manifold calculus of functors and embeddings

The Taylor tower also recovers some classical results in this context.

For example, the beginning of the tower, along with the connectivities of the maps, is



So if $n - 2m - 1 \geq 0$, then $\mathrm{Emb}(M, N) \to \mathrm{Imm}(M, N)$ is in particular surjective on π_0, which means that every immersion is homotopic (through immersions) to an embedding.

This is *Whitney's Easy Embedding Theorem*.

2(a). Manifold calculus of functors and embeddings

$$\mathrm{Emb}(M,N) \xrightarrow{2n-3m-3} T_2\,\mathrm{Emb}(M,N) \simeq \operatorname{holim} \left(\begin{array}{c} \mathrm{Imm}(M,N) \\ \downarrow \\ \mathrm{Map}^{\Sigma_2}(M\times M, N\times N) \\ \uparrow \\ \mathrm{ivmap}^{\Sigma_2}(M\times M, N\times N) \end{array} \right)$$

If $2n-3m-3 \geq 0$, then this map is surjective on π_0, and this is the *Haefliger/Dax Theorem.*

If $N=\mathbb{R}^n$, then $\mathrm{Map}(M,N) \simeq \mathrm{Map}^{\Sigma_2}(M\times M, N\times N) \simeq *$, and we have

Theorem (Dax)

$$\mathrm{Emb}(M,\mathbb{R}^n) \longrightarrow \mathrm{ivmap}^{\Sigma_2}(M\times M, \mathbb{R}^n\times\mathbb{R}^n)$$

is $(2n-3m-3)$*-connected.*

2(a). Manifold calculus of functors and embeddings

So when $2n - 3m - 3 \geq 0$ (*metastable range*), the problem of turning an immersion into an embedding is equivalent to studying the existence of liftings of elements of $T_1 \operatorname{Emb}(M, N)$ to $T_2 \operatorname{Emb}(M, N)$, i.e. liftings of immersions to isovariant maps.

It turns out that the obstruction to doing this is precisely the Haefliger's familiar *double point obstruction*: If the double points are null-cobordant, then the immersion is homotopic to an embedding (this can be stated in terms of some cohomology class vanishing).

The next natural thing to consider is the map

$$\operatorname{Emb}(M, N) \longrightarrow T_3 \operatorname{Emb}(M, N)$$

which can produce embeddings when $3n - 4m - 5 \geq 0$ (the $3/4$ *range*). The obstruction for turning an immersion into an embedding in this range, in addition to double points, is a manifold constructed out of triple points and some other data (Munson). If this manifold is null-cobordant, then an immersion can be turned into an embedding.

The pattern continues up the tower: $T_k \operatorname{Emb}(M, N)$ keeps track of k-fold self-intersections (and some other data).

2(b). Manifold calculus and the Tverberg Conjecture

This should be reminiscent of what happens in the

Tverberg Conjecture

Let $k \geq 2$, $n \geq 1$. Then any map $f \colon \Delta^{(k-1)(n+1)} \to \mathbb{R}^n$ maps points from k disjoint faces to the same point.

Recently showed false by Frick. Uses work of Mabillard and Wagner on the generalized Whitney trick which gives a way of resolving self-intersection points of maps of higher mutiplicities and formulates obstructions for doing this.

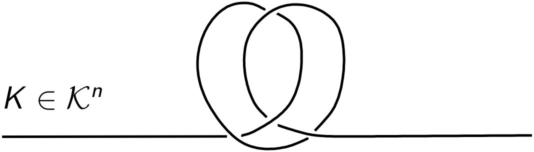
From manifold calculus point of view, this could be recast as:

- Consider the Taylor tower for $\mathrm{InjMap}(\Delta^{(k-1)(n+1)}, \mathbb{R}^n)$ (or some related mapping space/functor);
- The tower starts with $\mathrm{Map}(\Delta^{(k-1)(n+1)}, \mathbb{R}^n)$. Study the obstructions for lifting a map up the tower;
- Tverberg Conjecture is then about studying lifts from T_{k-1} to T_k.

Problem: Design the functor so as to account for the self-intersections coming from disjoint faces.

3. Space of long knots in $\mathbb{R}^n$, $n \geq 3$

$$\mathcal{K}^n = \{\text{embeddings } K : \mathbb{R} \hookrightarrow \mathbb{R}^n, \text{ fixed outside a compact set}\}$$
$$= \textit{space of long knots}$$



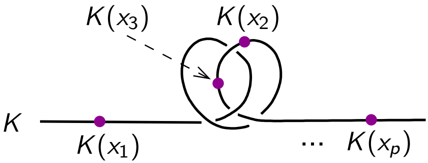
Classical knot theory is concerned with computing

- $H_0(\mathcal{K}^3)$, which is generated (over $\mathbb{R}$, say) by knot types, i.e. by isotopy classes of knots; and
- $H^0(\mathcal{K}^3)$, the set of knot invariants, i.e. locally constant ($\mathbb{R}$-valued) functions on $\mathcal{K}^3$, i.e. functions that take the same value on isotopic knots.

However, higher (co)homology and homotopy are also interesting, even when $n > 3$ (even though H^0 and H_0 are trivial in this case).

3(a). Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

To construct $T_k\mathcal{K}^n$: Can "sample" the knot $K : \mathbb{R} \hookrightarrow \mathbb{R}^n$ at p points $x_1, x_2, \ldots, x_p \in \mathbb{R}$.



The space of such samples is the configuration space $\text{Conf}(p, \mathbb{R}^n)$.

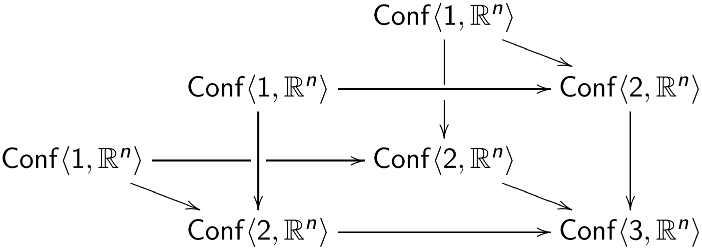
We can do this for all $1 \leq p \leq k$, and we can also relate the (compactified) configuration spaces via *doubling* maps ($p + 2$ of them for each p)

$$\text{Conf}\langle p, \mathbb{R}^n \rangle \longrightarrow \text{Conf}\langle p+1, \mathbb{R}^n \rangle, \quad 1 \leq p < k.$$

Together, these spaces and maps form a *punctured cubical diagram*.

3(a). Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

For example, when $k = 3$, we get

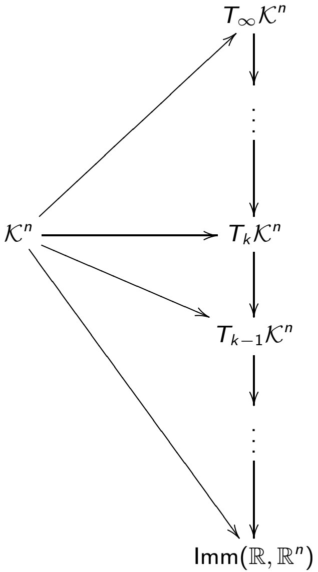


$T_3\mathcal{K}^n$ = homotopy limit (or homotopy pullback) of this diagram

This is analogous to the approximation of a function where we take k of its values and then form the interpolation polynomial.

There are also maps $\mathcal{K}^n \to T_k\mathcal{K}^n$ and $T_k\mathcal{K}^n \to T_{k-1}\mathcal{K}^n$ that are not hard to define, so that is how we get the Taylor tower for $\mathcal{K}^n$, $n \geq 3$:

3(a). Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$



Recall that, by Goodwillie-Klein-Weiss, this tower converges on homotopy and (co)homology for $n \geq 4$.

3(b). Application to homology of $\mathcal{K}^n$, $n \geq 4$

We have the Bousfield-Kan homology spectral sequence for the Taylor tower for $\mathcal{K}^n$, $n \geq 3$. It starts with

$$E^1_{-p,q} = \mathrm{H}_q(\mathrm{Conf}(p, \mathbb{R}^n)).$$

For $n \geq 4$, this spectral sequence converges to $\mathrm{H}_*(T_\infty \mathcal{K}^n)$, and hence to $\mathrm{H}_*(\mathcal{K}^n)$ by Goodwillie-Klein-Weiss.

Theorem (Lambrechts-Turchin-V. for $n \geq 4$, Kontsevich/V. for $n = 3$ on the diagonal, Moriya/Songhafouo-Tsopméné for $n = 3$ everywhere)

This homology spectral sequence collapses rationally at the E^2 page for $n \geq 3$.

Main ingredient in the proof: Kontsevich's rational formality of the little n-discs operad – this is the statement that chains on the operad are quasi-isomorphic to its homology – (plus model category techniques for $n = 3$).

(Collapse also true for the *homotopy* spectral sequence for $n \geq 4$; this is due to Arone-Lambrechts-Turchin-V.)

3(b). Application to homology of $\mathcal{K}^n$, $n \geq 4$

So for $n \geq 4$, the homology of the E^2 page is the homology of $\mathcal{K}^n$.
The main point:

> $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$ is built out of $\mathrm{H}_*(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{Q})$, which is understood and can be represented combinatorially with graph complexes. So we have a combinatorial description of $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$, $n \geq 4$.

Can in theory compute any homology group of any knot space for $n \geq 4$, and a lot is understood about the structure of $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$:

- it can be expressed in terms of the homology of the little n-cubes operad;
- it is a graded polynomial bialgebra generated by $\pi_*(\mathcal{K}^n) \otimes \mathbb{Q}$ (Lambrechts-Turchin);
- it admits a certain "Hodge decomposition" (Turchin);
- its rank has at least exponential growth (Turchin);
- it can be used to describe $\mathrm{H}_*(\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n); \mathbb{Q})$ (Arone-Turchin).

But we do not have a closed form expression for the homology.

4. Spaces of links

Let $n \geq 3$ and $m \geq 1$. Define

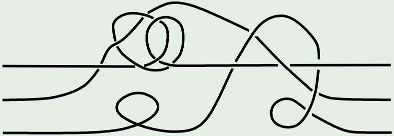
$$\mathcal{L}_m^n = \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\}$$
$$= \textit{space of long (string) links}$$

$$\mathcal{H}_m^n = \{\text{link maps} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\}$$
$$= \textit{space of homotopy long (string) links}$$

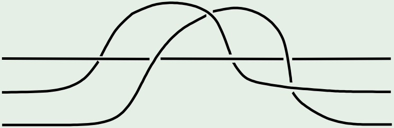
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint;
- All maps are standard outside a compact set;
- $\mathcal{L}_m^n \subset \mathcal{H}_m^n$.

Example

$H \in \mathcal{H}_3^n$



$L \in \mathcal{L}_3^n \subset \mathcal{H}_3^n$



In $\pi_0(\mathcal{H}_m^n)$, can pass a strand through itself so this can be thought of as space of "links without knotting".

4(a). Generalizations of results for knots to links

Can use *multivariable manifold calculus* (Munson-V.) to get, for $n \geq 3$,

- m-dimensional Taylor towers for $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$. The tower for $\mathcal{L}^n_m$ converges to this space for $n \geq 4$ (Munson-V.);
- Bousfield-Kan homology (and homotopy) spectral sequences for both towers;
- for $n \geq 4$, these spectral sequences start with
  - $\mathrm{H}_*(\mathrm{Conf}(km, \mathbb{R}^n))$, $k \geq 0$, for tower for $\mathcal{L}^n_m$;
  - $\mathrm{H}_*(\mathrm{Conf}(k, k, ..., k; \mathbb{R}^n))$, $k \geq 0$, for tower for $\mathcal{H}^n_m$.

Theorem (Munson-V.)

For $n \geq 4$, these homology spectral sequences converge to the inverse limits of their Taylor towers. The spectral sequence for the tower modeling links hence converges to $\mathcal{L}^n_m$ for $n \geq 4$.

(Can also do all this for braids, but not much interesting happens.)

4(a). Generalizations of results for knots to links

For links $\mathcal{L}_m^n$, the story is much like that for knots:

Theorem (Songhafouo-Tsopméné)

For $n > 5$, the rational homology spectral sequence for $\mathcal{L}_m^n$ collapses at the E^2 page.

- Proof again uses the formality of the little n-discs operad;
- As a consequence, we get a combinatorial description, via graph complexes, of the rational homology of $\mathcal{L}_m^n$, $n > 5$;
- This homology grows exponentially (Lambrechts-Komawila);
- Interesting connection to Sterling numbers (Lambrechts-Komawila).

4(b). Homotopy string links and subspace arrangements

For homotopy links $\mathcal{H}_m^n$, we know nothing. Not only do we not know whether the spectral sequence converges, but we also do not know if the Taylor tower converges to $\mathcal{H}_m^n$. So the spectral sequence is two steps removed from $\mathcal{H}_m^n$.

But $\mathcal{H}_m^n$ is a very interesting space since, for example, in the classical case $n = 3$, they are separated by *Milnor invariants* (Habegger-Lin). On the other hand, Milnor invariants factor through the Taylor tower for $\mathcal{H}_m^3$. In fact, this leads to

Theorem (Komendarczyk, Koytcheff, V.)

There is a correspondence (given by configuration space integrals) between Milnor invariants and trivalent trees.

The correspondence is only "up to lower order Milnor invariants", and we do not understand the combinatorics of trivalent trees well enough to remove this condition.

$\mathcal{H}_m^n$ is also interesting since it is one of the spaces of link maps $\mathrm{LkMap}(\sqcup_m S^k, S^n)$ studied by Koschorke, and his work has an interpretation in terms of manifold calculus (Munson).

4(b). Homotopy string links and subspace arrangements

So it is worth studying the "partial configuration spaces" or complements of subspace arrangements

$$\mathrm{Conf}(k, k, \dots, k; \mathbb{R}^n).$$

More generally, it is worth studying spaces

$$\mathrm{Conf}(k_1, k_2, \dots, k_m; \mathbb{R}^n) = \{(x_1^1, x_2^1, \dots, x_{k_1}^1, \dots, x_1^m, x_2^m, \dots, x_{k_m}^m) \in (\mathbb{R}^n)^{\sum k_i}$$
$$\text{such that } x_a^i \neq x_b^j \text{ for } i \neq j\}.$$

It seems that little is known about these spaces. We know their stable homotopy type (Ziegler-Živaljević) as the Spanier-Whitehead dual of

$$\bigvee_{p \in P} \left(\Delta(P_{<p}) * S^{d(p)-1}\right)$$

where P is the partition poset associated to the arrangement of the diagonals that have been removed and $d(p)$ is the dimension of the subspace corresponding to $p \in P$.

This can be used for computing the cohomology ring of these spaces (de Longueville-Schultz). We want their formality!

Further work

Further questions:

- Set up the right framework for studying the Tverberg conjecture in terms of manifold calculus of functors;
- Show formality of partial configuration spaces and the collapse of the spectral sequence for $\mathcal{H}^n_m$, $n \geq 4$;
- Show that the Taylor tower for $\mathcal{H}^n_m$, $n \geq 4$, converges;
- Figure out what the Taylor tower for all the spaces mentioned in this talk (including knots) converge to for $n = 3$;
- Reprove, in the setting of Taylor towers, that finite type invariants separate homotopy string links (Habegger-Lin);
- See if this helps in proving the same result for knots and links;
- Further explore the connection between manifold calculus and Koschorke's work on Milnor invariants of link maps of spheres.

Thank you!