Configuration space integrals, calculus of functors, and spaces of knots and links

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Plan of the talk

Part I: Configuration space integrals

1. Spaces of embeddings and long knots in particular
2. Configuration space integrals for knots in $\mathbb{R}^3$
3. Configuration space integrals for knots in $\mathbb{R}^n$, $n > 3$
4. Configuration space integrals for links

Part II: Manifold calculus of functors

1. General theory
2. Taylor tower for the space of long knots
3. Cosimplicial model for the Taylor tower
4. Applications to knot homology and finite type knot invariants
5. Multi-cosimplicial model for links, homotopy links, and braids
6. Applications to link homology and finite type link invariants

I.1. Spaces of embeddings

Definition

Let M and N be smooth manifolds. An *embedding* of M in N is an injective map $f\colon M \hookrightarrow N$ whose derivative is injective and which is a homeomorphism onto its image.

When M is compact, an embedding is an injective map with injective derivative.

The set of all embeddings of M in N can be topologized so we get the *space of embeddings* $\mathrm{Emb}(M, N)$. [1]

For many M and N, this is a topologically interesting space, so we want to know

$$\pi_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}^*(\mathrm{Emb}(M, N)).$$

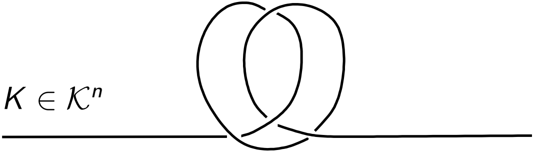
In particular, we have space of *long knots*:

[1] In practice, we actually take the homotopy fiber of the inclusion $\{\text{embeddings}\} \hookrightarrow \{\text{immersions}\}$.

I.1. Space of long knots in $\mathbb{R}^n$, $n \geq 3$

$$\mathcal{K}^n = \{\text{embeddings } K\colon \mathbb{R} \hookrightarrow \mathbb{R}^n, \text{ fixed outside a compact set}\}$$
$$= \textit{space of long knots}$$

$K \in \mathcal{K}^n$



Classical knot theory is concerned with computing

- $H_0(\mathcal{K}^3)$, which is generated (over $\mathbb{R}$, say) by knot types, i.e. by isotopy classes of knots (*isotopy* is homotopy in the space of embeddings); and
- $H^0(\mathcal{K}^3)$, the set of knot invariants, i.e. locally constant ($\mathbb{R}$-valued) functions on $\mathcal{K}^3$, i.e. functions that take the same value on isotopic knots.

However, higher (co)homology and homotopy are also interesting, even when $n > 3$ (even though H^0 and H_0 are trivial in this case).

I.2. Configuration space integrals: linking number

Related to the space of classical knots $\mathcal{K}^3$ is

$$\mathcal{L}^3_2 = \{\text{embeddings } \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}^3, \text{ fixed outside a compact set}\}$$
$$= \textit{space of long (string) links of two components}$$

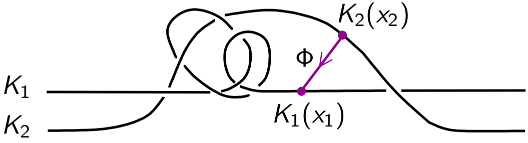
Define

$$\mathrm{Conf}(p, \mathbb{R}^n) = \{(x_1, x_2, \ldots, x_p) \in (\mathbb{R}^n)^p \colon x_i \neq x_j \text{ for } i \neq j\}$$
$$= \textit{configuration space of } p \textit{ points in } \mathbb{R}^n$$

Consider the maps Φ and π:

$$\Phi \colon \mathbb{R} \times \mathbb{R} \times \mathcal{L}^3_2 \xrightarrow{\textit{evaluation}} \mathrm{Conf}(2, \mathbb{R}^3) \xrightarrow{\textit{direction}} S^2$$

$$(x_1, x_2, L = (K_1, K_2)) \longmapsto (K_1(x_1), K_2(x_2)) \longmapsto \frac{K_2(x_2) - K_1(x_1)}{|K_2(x_2) - K_1(x_1)|}$$



$$\pi \colon \mathbb{R} \times \mathbb{R} \times \mathcal{L}^3_2 \xrightarrow{\textit{projection}} \mathcal{L}^3_2 \quad \text{(trivial bundle)}$$

So have a diagram

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2^3 & \xrightarrow{\Phi} & S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{L}_2^3 & & \end{array}$$

which, on the complex of deRham cochains (differential forms), gives a diagram

$$\begin{array}{ccc} \Omega^*(\mathbb{R} \times \mathbb{R} \times \mathcal{L}_2^3) & \xleftarrow{\Phi^*} & \Omega^*(S^2) \\ \downarrow{\scriptstyle \pi_*} & & \\ \Omega^{*-2}(\mathcal{L}_2^3) & & \end{array}$$

Here Φ^* is the usual pullback and π_* is *integration along the fiber*, or *pushforward* – a way to create forms on the base space of a bundle from forms on the total space, shifted by the dimension of the fiber.

I.2. Configuration space integrals: linking number

Let $sym_{S^2} \in \Omega^2(S^2)$ be the unit volume form on S^2, i.e.

$$sym_{S^2} = \frac{x\,dydz - y\,dxdz + z\,dxdy}{4\pi(x^2+y^2+z^2)^{3/2}}$$

Let $\alpha = \Phi^*(sym_{S^2})$. Then the linking number is

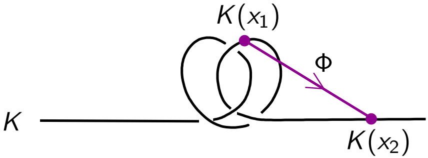
$$\boxed{Link(K_1, K_2) = \pi_*(\alpha) = \int_{\mathbb{R}\times\mathbb{R}} \alpha \in \Omega^0(\mathcal{L}_2^3)}$$

This is indeed a closed form, i.e. an element of $\mathrm{H}^0(\mathcal{L}_2^3)$, and hence an invariant of two-component links (this goes back to Gauss).

Now try to do the same, but for a single knot rather than a link.

I.2. Configuration space integrals: try to mimic $lk(K_1, K_2)$

The picture is



And the corresponding diagram is

$$\begin{array}{ccc} \mathrm{Conf}(2,\mathbb{R}) \times \mathcal{K}^3 & \xrightarrow{\Phi} & S^2 \\ \downarrow \pi & & \\ \mathcal{K}^3 & & \end{array}$$

The first issue is that an integral over $\mathrm{Conf}(2,\mathbb{R})$ may not converge since this space is open. So we compactify:

I.2. Configuration space integrals: Fulton-MacPherson compactification

Definition

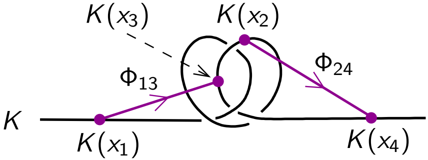
Let $\mathrm{Conf}[k,\mathbb{R}^n]$ be the *Fulton-MacPherson compactification* of $\mathrm{Conf}(k,\mathbb{R}^n)$.

Some properties:

- $\mathrm{Conf}[k,\mathbb{R}^n]$ is homotopy equivalent to $\mathrm{Conf}(k,\mathbb{R}^n)$;
- $\mathrm{Conf}[k,\mathbb{R}^n]$ is a manifold with corners;
- Boundary of $\mathrm{Conf}[k,\mathbb{R}^n]$ is characterized by points colliding with directions and relative rates of collisions kept track of;
- Stratification of the boundary given by stages of collisions of points; this stratification is encoded by trees;
- Works for configurations in any manifold, not just $\mathbb{R}^n$.

I.2. Configuration space integrals: simplest case for knots

But, even after compactifying, we still do not get an invariant. The next case is that of four points and two directions:



The maps are

$$\begin{array}{ccc} \mathrm{Conf}[4,\mathbb{R}] \times \mathcal{K}^3 & \xrightarrow{\Phi=\Phi_{13}\times\Phi_{24}} & S^2 \times S^2 \\ \downarrow \pi & & \\ \mathcal{K}^3 & & \end{array}$$

Let $\alpha = \Phi^*(sym^2_{S^2})$. Since α and $\mathrm{Conf}[4,\mathbb{R}]$, the fiber of π, are both 4-dimensional, we get a 0-form

$$I(\;$$  $$,K) = \pi_*(\alpha) = \int_{\mathrm{Conf}[4,\mathbb{R}]} \alpha$$

I.2. Configuration space integrals: simplest case for knots

So $I($$, K)$ is a 0-form, i.e. an element of $\Omega^0(\mathcal{K}^3)$. But is it a closed form, that is, is it an element of $\mathrm{H}^0(\mathcal{K}^3)$ – an invariant?

Want $dI($$, K) = 0$. Stokes' Theorem says that

$$dI(\,\cdot\,, K) = \pi_*(d\alpha) + (\partial\pi)_*(\alpha)$$

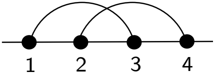
$$= (\partial\pi)_*(\alpha) \quad (\pi_*(d\alpha) = 0 \text{ since } sym_{S^2} \text{ is closed})$$

Here $(\partial\pi)_*(\alpha)$ is the pushforward along codimension one faces of $\mathrm{Conf}[4, \mathbb{R}]$.

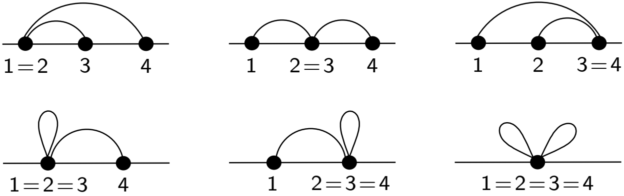
These faces can be represented by *diagrams* as follows.

I.2. Configuration space integrals: boundary diagrams

If there are four points moving on the knot, and two directions are kept track of as above, the diagram encoding this information is



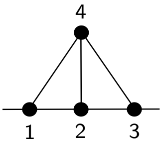
Codimension one faces (collisions of points) are then encoded by diagrams obtained from the above one by contracting segments between points (this mimics collisions)



(Loop corresponds to the derivative map.)
It turns out that the integrals corresponding to the bottom three diagrams vanish, but not necessarily for the top three.

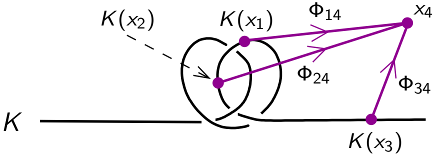
One way to resolve this: Look for another space to integrate over which has the same three faces and subtract the integrals.

The diagram that fits what we need is



since, when we contract edges to get $4=1$, $4=2$, and $4=3$, we get the same three relevant pictures as before (up to relabeling).

This suggests that we want a space of four configuration points in $\mathbb{R}^3$, three of which lie on a knot. In other words, we want the following picture:



I.2. Configuration space integrals: the fix

To make this precise, it turns out we can define a bundle (Bott-Taubes)

$$\pi\colon \operatorname{Conf}[3,1;\mathcal{K}^3,\mathbb{R}^3] \to \mathcal{K}^3$$

whose fiber over $K \in \mathcal{K}^3$ is the configuration space of four points, three of which are constrained to lie on K.

Let

$$\Phi = \Phi_{14} \times \Phi_{24} \times \Phi_{34}\colon \operatorname{Conf}[3,1;\mathcal{K}^3,\mathbb{R}^3] \longrightarrow (S^2)^3$$

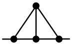
be the map giving the three directions as in the previous picture.

So the relevant maps are

$$\begin{array}{ccc} \operatorname{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \xrightarrow{\Phi} & (S^2)^3 \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{K} & & \end{array}$$

I.2. Configuration space integrals: the fix

Let $\alpha' = \Phi^*(sym^3_{S^2})$. This form can be integrated along the fiber $\mathrm{Conf}[3,1;K,\mathbb{R}^3]$ over K. Thus for each $K \in \mathcal{K}^3$, we get an integral

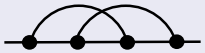
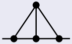
$$I(\ $$$$, K) = \pi_*(\alpha') = \int_{\mathrm{Conf}[3,1;K,\mathbb{R}^3]} \alpha'$$

It turns out that the boundary contributions for this integral are zero except for the three boundary pieces we care about. So we get

Theorem (Altschuler-Friedel, Bar-Natan)

The map

$$\mathcal{K} \longrightarrow \mathbb{R}$$

$$K \longmapsto \Big(I(\ $$$$, K) - I(\ $$$$, K) \Big)$$

is a knot invariant, i.e. an element of $\mathrm{H}^0(\mathcal{K}^3)$. *Further, it is a finite type two invariant.*

Digression on finite type invariants

Definition

A knot invariant is *finite type k* if it vanishes on the signed sums of resolutions of knots with k self-intersections.

Finite type invariants have received much attention in the last 15 years:

- Motivated by physics (Chern-Simons Theory);
- Connected to Lie algebras, three-manifold topology, etc.

Conjecture

The set of finite type invariants is a complete set of invariants.

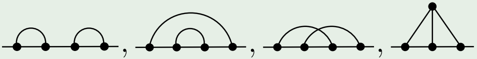
Getting back to configuration space integrals, there is no reason to stop at four configuration points:

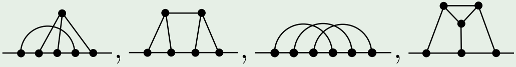
I.2. Configuration space integrals: general case

$\mathcal{TD}_k =$ {$\mathbb{R}$-vect. sp. gen'd by *trivalent diagrams* with $2k$ vertices, modulo *STU* and *IHX* relations}.

(STU and IHX are some relations on the vector space of diagrams.)

Example

$\mathcal{TD}_2 = \{$  $\}$

$\mathcal{TD}_3 = \{$  , etc.$\}$

I.2. Configuration space integrals: general case

Given $D \in \mathcal{TD}_k$ with p vertices on the segment and q off the segment, again have a bundle

$$\mathrm{Conf}[p, q; \mathcal{K}^3, \mathbb{R}^3] \longrightarrow \mathcal{K}^3$$

Also have map

$$\Phi\colon\ \mathrm{Conf}[p, q; \mathcal{K}^3, \mathbb{R}^3] \longrightarrow (S^2)^e$$

where

- Φ is the product of the direction maps between pairs of configuration points corresponding to the edges of D, and
- e is the number of edges of D.

Let $\alpha = \Phi^*(sym^e_{S^2})$.

Then for each $K \in \mathcal{K}^3$, have integral

$$\boxed{I(D, K) = \pi_*(\alpha) = \int_{\mathrm{Conf}[p,q;K,\mathbb{R}^3]} \alpha}$$

I.2. Configuration space integrals: general case

Let $\mathcal{W}_k = \mathcal{TD}_k^*$ (space of *weight systems*).

Theorem (Bott-Taubes, D.Thurston)

For each $W \in \mathcal{W}_k$, the map

$$\begin{aligned} \mathcal{K}^3 &\longrightarrow \mathbb{R} \\ K &\longmapsto \sum_{D \in \mathcal{TD}_k} W(D) I(D, K) \end{aligned}$$

is a knot invariant (up to the anomalous correction). Further, it is a finite type k invariant. In fact, we have an isomorphism

$$\mathcal{W}_k \xrightarrow[\cong]{\text{configuration space integrals}} \{\text{finite type } k \text{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}^3).$$

This theorem is also called the *Fundamental Theorem of Finite Type (or Vassiliev) Invariants* and was first proved by Kontsevich using the famous *Kontsevich Integral*

I.3. Generalization to knot spaces in dimension > 3

Recall that for $\mathcal{K}^n$, $n \geq 4$, we are interested in understanding its topology, namely we would like to know

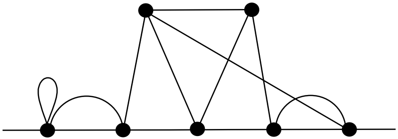
$$\mathrm{H}_*(\mathcal{K}^n) \quad \text{and} \quad \mathrm{H}^*(\mathcal{K}^n).$$

$\mathrm{H}^0(\mathcal{K}^n)$ and $\mathrm{H}_0(\mathcal{K}^n)$ are now trivial, but higher (co)homology is very interesting.

Idea: produce cohomology classes of $\mathcal{K}^n$ in various degrees (and not just in degree 0) using configuration space integrals.

I.3. Generalization to knot spaces in dimension > 3

Take more general diagrams (*at least* trivalent), such as



For each $n \geq 3$, let

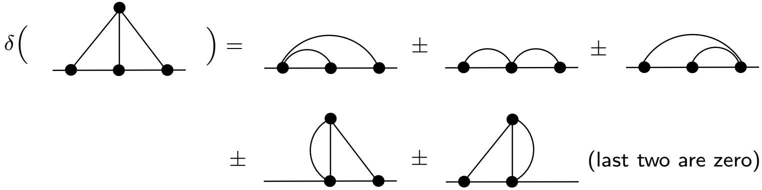
$$\mathcal{D}^n = \{\mathbb{R}\text{-vect. sp. gen'd by diagrams with valence} \geq 3\},$$

where diagrams are connected, vertices are labeled, no loops on off-segment vertices, edges are labeled or oriented (depending on parity of n). Mod out by diagrams with double edges and impose some sign relations.

Degree of $D \in \mathcal{D}$ is

$$\deg(D) = 2(\#\text{edges}) - 3(\#\text{off-segment vert.}) - (\#\text{segment vert.})$$

Coboundary δ is given by contracting non-chord and non-loop edges and segments, for example



(last two are zero)

Easy to see that δ raises degree by 1 and that $\delta^2 = 0$. Thus

$(\mathcal{D}^n, \delta)$ is a cochain complex.

I.3. Generalization to knot spaces in dimension > 3

For each $D \in \mathcal{D}^n$ and $K \in \mathcal{K}^n$, we can still set up the integral $I(D, K)$ as before. The only difference is that we will not necessarily get a form in degree zero but in some degree of $\Omega^*(\mathcal{K}^n)$.

Theorem (Cattaneo, Cotta-Ramusino, Longoni)

For $n > 3$, configuration space integrals give a cochain map

$$I_{\mathcal{K}} : (\mathcal{D}^n, \delta) \longrightarrow (\Omega^*(\mathcal{K}^n), d).$$

Corollary

The knot space $\mathcal{K}^n$, $n > 3$, has nontrivial cohomology beyond arbitrarily high dimension.

Conjecture

This map is a quasi-isomorphism.

This is compatible with what we already did in the case of classical knots $\mathcal{K}^3$:

For $n = 3$, one does not get a cochain map in all degrees, but in degree zero the map can be modified so that it does commute with the differential. So we can see what happens on H^0. It turns out that

$$\mathrm{H}^0(\mathcal{D}^3) = \mathcal{TD} \quad \text{(trivalent diagrams)}$$

So kernel of δ in degree zero is defined by imposing the the STU and IHX relations. Thus we get a map (after identifying $\mathcal{TD}$ with its dual, the weight systems $\mathcal{W}$),

$$(\mathrm{H}^0(\mathcal{D}^3))^* = \mathcal{W} \longrightarrow \mathrm{H}^0(\mathcal{K}).$$

But we already know that the image of this map is precisely the finite type knot invariants.

I.4. Configuration space integrals for spaces of links

Let $n \geq 3$ and $m \geq 1$. Related to $\mathcal{K}^n$ are

$$\begin{aligned}
\mathcal{L}_m^n &= \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of long (string) links} \\
\mathcal{H}_m^n &= \{\text{link maps} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of homotopy long (string) links} \\
\mathcal{B}_m^n &= \{\text{embeddings with positive derivative} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of pure braids}
\end{aligned}$$

- All maps are standard outside a compact set;
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint.

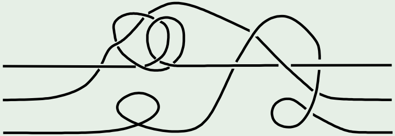
(As with knots, we in practice work with the homotopy fiber of the inclusion embeddings $\hookrightarrow$ immersions for $\mathcal{L}_m^n$ and $\mathcal{B}_m^n$.)

I.4. Configuration space integrals for spaces of links

- $\mathcal{B}^n_m \subset \mathcal{L}^n_m \subset \mathcal{H}^n_m$;
- In $\pi_0(\mathcal{H}^n_m)$, can pass a strand through itself so this can be thought of as space of "links without knotting".

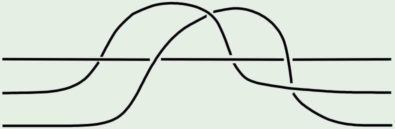
Example

$H \in \mathcal{H}^n_3$



$\leftarrow K \in \mathcal{K}^n$

$L \in \mathcal{B}^n_3 \subset \mathcal{L}^n_3 \subset \mathcal{H}^n_3$



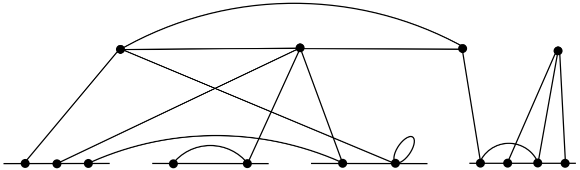
I.4. Configuration space integrals for spaces of links

Recall the cochain map

$$I_{\mathcal{K}} \colon \mathcal{D}^n \longrightarrow \Omega^*(\mathcal{K}^n)$$

Generalize the diagram complex $\mathcal{D}^n$ to a complex $\mathcal{LD}^n_m$ and a subcomplex $\mathcal{HD}^n_m$.

$\mathcal{LD}^n_m$ is defined the same way as $\mathcal{D}^n$ except there are now m segments, e.g.



$\in \mathcal{LD}^n_4$

$\mathcal{HD}^n_m$ is defined by imposing: If there exists a path between distinct vertices on a given segment, then it must pass through a vertex on another segment.

I.4. Configuration space integrals for spaces of links

Theorem (Koytcheff, Munson, V.)

There are integration maps $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{HD}_m^n & \xrightarrow{I_{\mathcal{H}}} & \Omega^*(\mathcal{H}_m^n) \\ \big\downarrow & & \big\downarrow \\ \mathcal{LD}_m^n & \xrightarrow{I_{\mathcal{L}}} & \Omega^*(\mathcal{L}_m^n) \end{array}$$

$I_{\mathcal{L}}$ is a cochain map for $n > 3$ and $I_{\mathcal{H}}$ is a cochain map for $n \geq 3$. Further, for $n = 3$, we have isomorphisms

$$(\mathrm{H}^0(\mathcal{LD}_m^3))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{L}_m^3\} \in \mathrm{H}^0(\mathcal{L}_m^3)$$

$$(\mathrm{H}^0(\mathcal{HD}_m^3))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{H}_m^3\} \in \mathrm{H}^0(\mathcal{H}_m^3)$$

Conjecture

$I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are quasi-isomorphisms for $n > 3$ and $n \geq 3$, respectively.

I.4. Configuration space integrals for spaces of links

It is known that *Milnor invariants* of long homotopy links are finite type invariants. Thus get

Corollary

The map $I_{\mathcal{H}}$ provides configuration space integral expressions for Milnor invariants of $\mathcal{H}^3_m$.

Remarks:

1. It is somewhat surprising that configuration space integrals can be defined for homotopy links.
2. Can do all this for braids as well. One should be able to connect to work of T. Kohno on braids and Chen integrals.

Generalization to spaces of embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$

K. Sakai has recently done a lot of work on configuration space integrals:

- Produces a cohomology class of $\mathcal{K}^3$ in degree one that is related to the Casson invariant using integrals;
- (with Watanabe) There is a diagram complex $\mathcal{D}^{k,n}$ and a linear map
  $$\mathcal{D}^{k,n} \longrightarrow \mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$$
  which is a cochain map for some subcomplexes of $\mathcal{D}^{k,n}$ and certain parity conditions on k and n. Also have some non-triviality results;
- New interpretation of the Haefliger invariant for $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$ for some k and n.

(The story for knots and links connects to functor calculus, as we will see next, and it would be nice to do the same for this work).

II.1. General manifold calculus of functors

Let Top be the category of topological spaces and let

$\mathcal{O}(M)$ = category of open subsets of M with inclusions as morphisms.

Manifold calculus studies functors

$$F \colon \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}$$

One such functor is the space of embeddings $\mathrm{Emb}(-, N)$, where N is a smooth manifold, since, given an inclusion

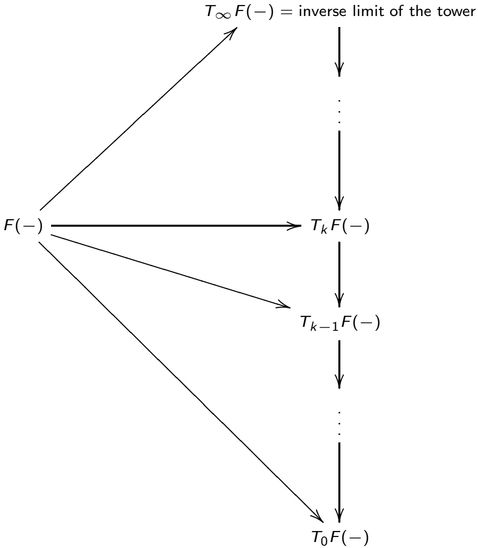
$$O_1 \hookrightarrow O_2$$

of open subsets of M, there is a restriction

$$\mathrm{Emb}(O_2, N) \to \mathrm{Emb}(O_1, N).$$

II.1. General manifold calculus of functors

For any functor $F\colon \mathcal{O}(M)^{op} \to \mathrm{Top}$, the theory produces a "Taylor tower" of approximating functors/fibrations



II.1. General manifold calculus of functors

Theorem (Goodwillie-Klein-Weiss)

For $F = \mathrm{Emb}(-, N)$ and for $4dim(M) \leq dim(N)$, the Taylor tower converges on (co)homology, i.e.

$$\mathrm{H}_*(\mathrm{Emb}(-, N)) \cong \mathrm{H}_*(T_\infty \mathrm{Emb}(-, N)).$$

In particular, evaluating at M gives

$$\mathrm{H}_*(\mathrm{Emb}(M, N)) \cong \mathrm{H}_*(T_\infty \mathrm{Emb}(M, N)).$$

For $dim(M) + 3 \leq dim(N)$, same is true for π_*.

Note that when M is 1-dimensional, N has to be at least 4-dimensional in both conditions.

Let's see how this theory applies in the case of long knots:

II.2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

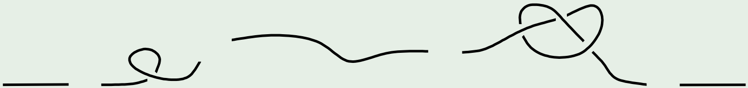
Remember that $\mathcal{K}^n$ is the space of long embeddings of $\mathbb{R}$ in $\mathbb{R}^n$.
To construct $T_k\mathcal{K}^n$, let $I_1, \ldots, I_{k+1}$ be disjoint subintervals of $\mathbb{R}$ and

$$\emptyset \neq S \subseteq \{1, \ldots, k+1\}.$$

Then let

$$\mathcal{K}^n_S = \mathrm{Emb}(\mathbb{R} \setminus \bigcup_{i \in S} I_i,\ \mathbb{R}^n) = \text{space of “punctured knots”}$$

Example



an element of $\mathcal{K}^n_{\{1,2,3,4\}}$

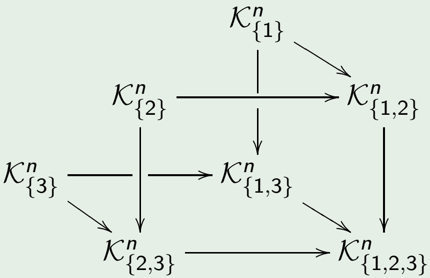
These spaces are not very interesting on their own, and are in fact connected even for $n = 3$. But...

II.2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

Have restriction maps $\mathcal{K}^n_S \to \mathcal{K}^n_{S \cup \{i\}}$ given by punching another hole. These spaces and maps then form a diagram of knots with holes (such a diagram is sometimes called a *punctured cube*).

Example

When $k = 2$, we get



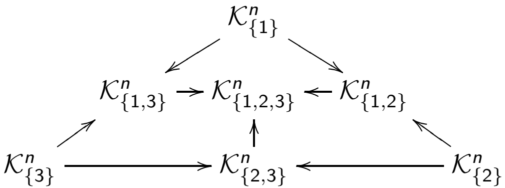
Definition

The kth stage of the Taylor tower for $\mathcal{K}^n$, $n \geq 3$, is the homotopy limit of the punctured cube from the previous slide. In other words,

$$T_k\mathcal{K}^n = \operatorname*{holim}_{\emptyset \neq S \subseteq \{1,..,k+1\}} \mathcal{K}^n_S.$$

Homotopy limit of a diagram should be thought of as the limit, namely the subspace of the product of the spaces in the diagram consisting of points that are compatible with the maps in the diagram, but "fattened up" so that it is made homotopy invariant.

Not hard to see what this homotopy limit is: The punctured cubical diagram from before can be redrawn as



Then a point in $T_2\mathcal{K}^n$ is

- A point in each $\mathcal{K}^n_{\{i\}}$ (once-punctured knot);
- A path in each $\mathcal{K}^n_{\{i,j\}}$ (isotopy of a twice-punctured knot) ;
- A two-parameter path in $\mathcal{K}^n_{\{1,2,3\}}$ (two-parameter isotopy of a thrice-punctured knot); and
- Everything is compatible with the restriction maps.

There is a map

$$\mathcal{K}^n \longrightarrow T_k\mathcal{K}^n$$

given by punching holes in the knot (the isotopies in the homotopy limit are thus constant).

Easy to see: For $k \geq 3$, $\mathcal{K}^n$ is the actual pullback (limit) of the subcubical diagram.

So the strategy is to replace the limit, which is what we really care about, by the homotopy limit, which is hopefully easier to understand.

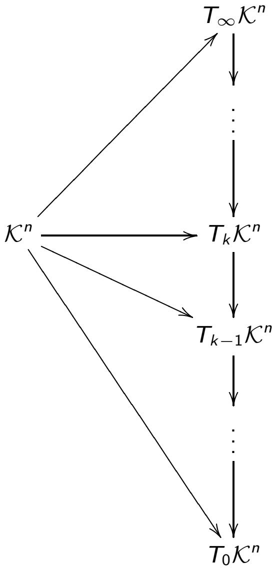
There is also a map, for all $k \geq 1$,

$$T_k\mathcal{K}^n \longrightarrow T_{k-1}\mathcal{K}^n,$$

since the diagram defining $T_{k-1}\mathcal{K}^n$ is a subdiagram of the one defining $T_k\mathcal{K}^n$ and hence the homotopy limit of the bigger diagram maps to the homotopy limit of the smaller one.

Putting these maps and spaces together, we get the Taylor tower for $\mathcal{K}^n$, $n \geq 3$:

II.2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

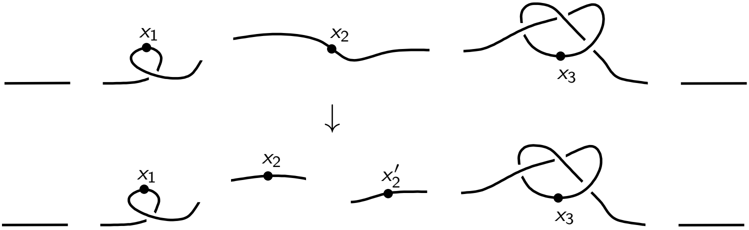


By Goodwillie-Klein-Weiss, this tower converges on homotopy and (co)homology for $n \geq 4$.

By retracting the arcs of a punctured knot, we get

$$\mathcal{K}_S^n \simeq \mathrm{Conf}(|S|-1, \mathbb{R}^n).$$

The restriction maps "add a point":



This is made precise with a *cosimplicial space*:

II.3. Cosimplicial model for the Taylor tower

Definition

Let $(K^n)^\bullet$ be the cosimplicial space

$$(K^n)^\bullet = \Big(\mathrm{Conf}\langle 0, \mathbb{R}^n\rangle \rightleftarrows \mathrm{Conf}\langle 1, \mathbb{R}^n\rangle \rightleftarrows \mathrm{Conf}\langle 2, \mathbb{R}^n\rangle \cdots \Big),$$

where $\mathrm{Conf}\langle p, \mathbb{R}^n\rangle$ is a variant of the Fulton-MacPherson compactification, cofaces $\xrightarrow{d^i}$ are doubling (diagonal) maps and codegeneracies $\xleftarrow{s^i}$ are forgetting maps.

Let $\mathrm{Tot}^k(K^n)^\bullet$ be the kth partial totalization of $(K^n)^\bullet$, i.e. the homotopy limit of the truncation of $(K^n)^\bullet$ at kth stage and let $\mathrm{Tot}(K^n)^\bullet$ the inverse limit of these partial totalizations.

Theorem (Sinha)

For $n \geq 2$ and $k \geq 0$, $\mathrm{Tot}^k(K^n)^\bullet \simeq T_k\mathcal{K}^n$ (and so $\mathrm{Tot}(K^n)^\bullet \simeq T_\infty\mathcal{K}^n$).

Combining this with Godwillie-Klein-Weiss, we get

Corollary

For $n \geq 4$, $\mathrm{Tot}(K^n)^\bullet \simeq \mathcal{K}^n$.

II.4. Application to homology of $\mathcal{K}^n$, $n \geq 4$

The reason we care about the cosimplicial model is that we have the Bousfield-Kan homology spectral sequence for $(K^n)^\bullet$, $n \geq 3$. It starts with

$$E^1_{-p,q} = \mathrm{H}_q(\mathrm{Conf}(p, \mathbb{R}^n)).$$

For $n \geq 4$, this spectral sequence converges to $\mathrm{H}_*(\mathrm{Tot}(K^n)^\bullet)$, and hence to $\mathrm{H}_*(\mathcal{K}^n)$ by Goodwillie-Klein-Weiss.

Theorem (Lambrechts-Turchin-V. for $n \geq 4$, Kontsevich/V. for $n = 3$ on the diagonal, Moriya/Songhafouo-Tsopméné for $n = 3$ everywhere)

This homology spectral sequence collapses rationally at the E^2 page for $n \geq 3$.

Main ingredient in the proof: Kontsevich's rational formality of the little n-cubes operad (plus model category techniques for $n = 3$). Key step in formality – configuration space integrals!

(Collapse also true for the *homotopy* spectral sequence for $n \geq 4$; this is due to Arone-Lambrechts-Turchin-V.)

II.4. Application to homology of $\mathcal{K}^n$, $n \geq 4$

So for $n \geq 4$, the homology of the E^2 page is the homology of $\mathcal{K}^n$. Fancier way to say this, using the fact that $(K^\bullet)^n$ also comes from the *little cubes operad*, is

Theorem

For $n \geq 4$, $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q}) \cong \mathrm{HH}_*(\mathcal{POISS}_{n-1})$, where $\mathcal{POISS}_{n-1}$ is the operad obtained by taking the homology of the little n-cubes operad.

The main point:

$\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$ is built out of $\mathrm{H}_*(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{Q})$, which is understood. In fact, it can be represented combinatorially with graph complexes. So we have a nice combinatorial description of $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$, $n \geq 4$.

The cohomological version is

Corollary

The differential graded algebra model for $\mathrm{H}^*(\mathcal{K}^n; \mathbb{Q})$, $n \geq 4$, is

$$\left(\bigoplus_{p=0}^{\infty} s^{-p}\, \mathrm{H}^*(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{Q}), \sum \pm \mathrm{H}^*(d^i) \right).$$

II.4. Application to finite type invariants of $\mathcal{K}^3$

Want to see what this machinery can say about knot invariants. The cohomology spectral sequence for $(K^3)^\bullet$ is *trying* to compute these in degree zero – we do not even know that it converges to $\mathrm{H}^0(\mathrm{Tot}(K^3)^\bullet)$, let alone to $\mathrm{H}^0(\mathcal{K}^3)$. More precisely, we just have maps

$$\bigoplus_p E_2^{-p,p} \longrightarrow \mathrm{H}^0(\mathrm{Tot}(K^3)^\bullet) \longrightarrow \mathrm{H}^0(\mathcal{K}^3),$$

where $E_2^{-p,p}$ is on the diagonal of the spectral sequence, i.e. in total degree 0. We do not know if these maps are isomorphisms.

It is not hard to see that $E_2^{-p,p}$ can be represented exactly by trivalent diagrams modulo STU and IHX relations from the theory of finite type knot invariants.

II.4. Application to finite type invariants of $\mathcal{K}^3$

Theorem (V.)

The Taylor tower for $\mathcal{K}^3$ classifies finite type knot invariants. More precisely, for each $k \geq 0$, there is an isomorphism (over $\mathbb{R}$)

$$\mathrm{H}^0(T_{2k}\mathcal{K}^3) \cong \{\textit{finite type } k \textit{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}^3).$$

(And $\mathrm{H}^0(T_{2k}\mathcal{K}^3) \cong \mathrm{H}^0(T_{2k+1}\mathcal{K}^3)$.)

Idea of proof.

Show that configuration space integrals factor through the Taylor tower and that all the maps are isomorphisms:

$$\begin{array}{ccc} \mathcal{W}_k & \xrightarrow[\cong]{\text{configuration space integrals}} & \{\text{fin. type } k \text{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}^3) \\ & {\searrow}^{\cong} \qquad\qquad {\nearrow}_{\cong} & \\ & \mathrm{H}^0(T_{2k}\mathcal{K}^3) \cong \mathrm{H}^0(T_{2k+1}\mathcal{K}^3) & \end{array}$$

□

II.5. Two-variable manifold calculus

If $M = P \coprod Q$, can apply two-variable calculus (Munson-V.) for contravariant functors F on $\mathcal{O}(P) \times \mathcal{O}(Q)$ (rather than on $\mathcal{O}(P \coprod Q)$). Get bitower

$$\begin{array}{ccccccc}
T_{0,\infty}F(-,-) & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & T_{\infty,\infty}F(-,-) \\
\downarrow & & & & & & \downarrow \\
\vdots & & & & & & \vdots \\
\downarrow & & & & & & \downarrow \\
T_{0,1}F(-,-) & \longleftarrow & T_{1,1}F(-,-) & \longleftarrow & \cdots & \longleftarrow & T_{\infty,1}F(-,-) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_{0,0}F(-,-) & \longleftarrow & T_{1,0}F(-,-) & \longleftarrow & \cdots & \longleftarrow & T_{\infty,0}F(-,-)
\end{array}$$

- Connection to single-variable calculus: $T_k F = \operatorname*{holim}_{k_1+k_2 \leq k} T_{k_1,k_2} F$;
- Have same convergence result as in ordinary manifold calculus: For $F = \mathrm{Emb}(P \coprod Q, N)$ and same dimensional assumptions, the bitower converges.

II.5. Multi-cosimplicial model for links

Let $I_1, \ldots, I_{k_1+1}$ be disjoint intervals in $\mathbb{R}$. Same for $J_1, \ldots, J_{k_2+1}$. Then

Definition

$$T_{k_1,k_2}\mathcal{L}_2^n = \underset{\substack{\emptyset \neq S_1 \subseteq \{1,\ldots,k_1\} \\ \emptyset \neq S_2 \subseteq \{1,\ldots,k_2\}}}{\operatorname{holim}} \operatorname{Emb}\Big((\mathbb{R} \setminus \bigcup_{i \in S_1} I_i) \coprod (\mathbb{R} \setminus \bigcup_{j \in S_j} J_j) \ , \ \mathbb{R}^n \Big).$$

Now get a bicosimplicial space $(L^n)^{\bullet,\bullet}$ whose (k_1, k_2) entry is $\mathrm{Conf}\langle k_1 + k_2, \mathbb{R}^n \rangle$ (can "double" and "forget" points in two directions).

Let $\mathrm{Tot}^{k_1,k_2}(L^n)^{\bullet,\bullet}$ be the (k_1, k_2)th partial totalization of $(L^n)^{\bullet,\bullet}$.

Proposition (Munson-V.)

For $n \geq 2$ and $k_1, k_2 > 0$, $\mathrm{Tot}^{k_1,k_2}(L^n)^{\bullet,\bullet} \simeq T_{k_1,k_2}\mathcal{L}_2^n$.

Want to associate a spectral sequence to this bicosimplicial space. To do this, first take the *diagonal cosimplicial space*

$$(L^n)^{\bullet,\bullet}_{diag} = \{\, \mathrm{Conf}\langle 2k, \mathbb{R}^n\rangle \}_{k=0}^{\infty}.$$

It is not hard to see that

$$\mathrm{Tot}(L^n)^{\bullet,\bullet} = \mathrm{Tot}(L^n)^{\bullet,\bullet}_{diag}.$$

As before, have the Bousfield-Kan homology spectral sequence for $(L^n)^{\bullet,\bullet}_{diag}$, $n \geq 3$, with

$$E^1_{-p,q} = \mathrm{H}_q(\mathrm{Conf}(2p, \mathbb{R}^n))$$

which, for $n \geq 4$, converges to

$$\mathrm{H}_*(\mathrm{Tot}(L^n)^{\bullet,\bullet}_{diag}) = \mathrm{H}_*(\mathcal{L}^n_2).$$

(Similar for the homotopy spectral sequence.)

II.5. Modifications for homotopy string links and braids

Recall the spaces $\mathcal{H}_2^n$ and $\mathcal{B}_2^n$ of homotopy string links and braids. Again have bicosimplicial models for the bitowers of punctured homotopy links and braids, as well as their diagonal cosimplicial spaces, except:

- For $\mathcal{H}_2^n$, the kth space in the diagonal cosimplicial space is

  $$\mathrm{Conf}\langle k,k;\,\mathbb{R}^n\rangle = \text{compactification of } \{(x_1,\ldots,x_k,y_1,\ldots,y_k)\in(\mathbb{R}^n)^{2k} : x_i\neq y_j\}$$

  This is a kind of a compactified "partial configuration space" or a complement of a hyperspace arrangement.

- For $\mathcal{B}_2^n$, the kth space in the diagonal cosimplicial space is

  $$(\mathrm{Conf}\langle 2,\mathbb{R}^{n-1}\rangle)^k$$

  and this turns out to give the standard cosimplicial model for $\Omega\,\mathrm{Conf}(2,\mathbb{R}^{n-1})$ (which is exactly what braids are).

II.5. Generalization to more than two strands

Generalization to m-component links is straightforward:
Get, for $n \geq 3$,

- m-dimensional Taylor towers for $\mathcal{L}^n_m$, $\mathcal{H}^n_m$, and $\mathcal{B}^n_m$;
- m-cosimplicial models for these towers;
- diagonal cosimplicial spaces
  - $(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ consisting of $\mathrm{Conf}\langle km, \mathbb{R}^n\rangle$, $k \geq 0$;
  - $(H^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ consisting of $\mathrm{Conf}\langle k, k, \ldots, k; \mathbb{R}^n\rangle$, $k \geq 0$; and
  - $(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ consisting of $(\mathrm{Conf}\langle m, \mathbb{R}^{n-1}\rangle)^k$, $k \geq 0$

  modeling the towers (so their totalizations are equivalent to inverse limits of the towers);
- Bousfield-Kan homology (and homotopy) spectral sequences for these cosimplicial spaces.
- for $n \geq 4$,

  $$\mathrm{H}_*(\mathrm{Tot}(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}) \cong \mathrm{H}_*(\mathcal{L}^n_m) \quad \text{and} \quad \mathrm{H}_*(\mathrm{Tot}(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}) \cong \mathrm{H}_*(\mathcal{B}^n_m)$$

  (same for π_*)

So how many of the results we had for knots carry over?

II.6. Application to link cohomology

Theorem (Munson-V.)

For $n \geq 4$, the homology spectral sequences for all three cosimplicial spaces converge to their totalization. The spectral sequences for $(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ and $(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ hence converge to $\mathcal{L}^n_m$ and $\mathcal{B}^n_m$.

- Do the rational homology (and homotopy) spectral sequences for $(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ and $(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ collapse at E^2 for $n \geq 3$ (in the spirit of Lambrechts-Turchin-V./Moriya/Songhafouo-Tsopméné)?

Consequence: A combinatorial description, via graph complexes, of rational homology of $\mathcal{L}^n_m$ and $\mathcal{B}^n_m$ (latter is already understood).

- Does the rational homology (and homotopy) spectral sequence for $(H^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ collapse at E_2 for $n \geq 3$?

This seems to be a harder problem, and solving it would *not necessarily* give a description of rational cohomology of $\mathcal{H}^n_m$ since we do not know if the Taylor tower for $\mathcal{H}^n_m$ converges.

But if we in particular had collapse on the diagonal for $n = 3$ for the three cosimplicial spaces, we would also have

Consequence: Taylor multi-towers classify finite type invariants of $\mathcal{L}^3_m$, $\mathcal{H}^3_m$, and $\mathcal{B}^3_m$.

(The connection between finite type invariants and Taylor multi-towers for these three spaces is given by configuration space integrals as in the case of knots.)

Further consequence: Since *Milnor invariants* of $\mathcal{H}^3_m$ are finite type, this would place these classical invariants in the context of manifold calculus of functors.

Further work

So the main message is

> Taylor towers contain information about topology of knots and links and configuration space integrals help us understand it.

Further questions:

- Show that the Taylor tower for $\mathcal{H}^n_m$, $n \geq 4$, converges;
- Figure out what the Taylor tower for all the spaces mentioned in this talk (including knots) converge to for $n = 3$ (group completion?);
- Reprove, in the setting of Taylor towers, that finite type invariants separate braids (Kohno, Bar-Natan) and homotopy string links (Habegger-Lin);
- See if this helps in proving the same result for knots and links;
- Generalize Milnor invariants to homotopy links of spheres (or planes) in any dimension and connect with work of Koschorke. Show, using manifold calculus, that these generalizations suffice for separation of link maps of spheres.
- Connect to work of Sakai and Sakai-Watanabe, as mentioned before.

Thank you!