Homotopy-theoretic methods in the study of spaces of knots and links

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Tokyo University Topology Seminar
December 11, 2012

Plan of the talk

Main point: *Manifold calculus of functors* can be used effectively to study algebraic topology of various knot and link spaces.

Outline:

Part I: Knots

1. Manifold calculus of functors
2. Taylor tower for the space of long knots
3. Cosimplicial model for the Taylor tower
4. Application to homology of the space of long knots
5. Application to finite type knot invariants

Part II: Links

6. Multivariable manifold calculus of functors
7. Multi-cosimplicial model for links, homotopy links, and braids
8. Applications to link cohomology and finite type link invariants

1. Manifold calculus of functors

Definition

Let M and N be smooth manifolds. An *embedding* of M in N is an injective map $f \colon M \hookrightarrow N$ whose derivative is injective and which is a homeomorphism onto its image.

When M is compact, an embedding is an injective map with injective derivative.

The set of all embeddings of M in N can be topologized so we get the *space of embeddings* $\mathrm{Emb}(M, N)$ (a special case of a *mapping space*). [1]

For many M and N, this is a topologically interesting space, so we want to know

$$\pi_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}_*(\mathrm{Emb}(M, N)), \quad \mathrm{H}^*(\mathrm{Emb}(M, N)).$$

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[1] In practice, we actually take the homotopy fiber of the inclusion $\{\text{embeddings}\} \hookrightarrow \{\text{immersions}\}$.

1. Manifold calculus of functors

Let Top be the category of topological spaces and let

$\mathcal{O}(M) =$ category of open subsets of M with inclusions as morphisms.

Manifold calculus studies functors

$$F\colon \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}$$

One such functor is the space of embeddings $\mathrm{Emb}(-, N)$, where N is a smooth manifold, since, given an inclusion

$$O_1 \hookrightarrow O_2$$

of open subsets of M, there is a restriction

$$\mathrm{Emb}(O_2, N) \to \mathrm{Emb}(O_1, N).$$

1. Manifold calculus of functors

For any functor $F\colon \mathcal{O}(M)^{op} \to \mathrm{Top}$, the theory produces a "Taylor tower" of approximating functors/fibrations

$$\begin{tikzcd}
 & T_\infty F(-) = \text{inverse limit of the tower} \arrow[d] \\
 & \vdots \arrow[d] \\
F(-) \arrow[ruu] \arrow[r] \arrow[rd] \arrow[rddd] & T_k F(-) \arrow[d] \\
 & T_{k-1} F(-) \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_0 F(-)
\end{tikzcd}$$

1. Manifold calculus of functors

Theorem (Goodwillie-Klein-Weiss)

For $F = \mathrm{Emb}(-, N)$ *and for* $4dim(M) \leq dim(N)$*, the Taylor tower converges on (co)homology, i.e.*

$$\mathrm{H}_*(\mathrm{Emb}(-, N)) \cong \mathrm{H}_*(T_\infty \, \mathrm{Emb}(-, N)).$$

In particular, evaluating at M *gives*

$$\mathrm{H}_*(\mathrm{Emb}(M, N)) \cong \mathrm{H}_*(T_\infty \, \mathrm{Emb}(M, N)).$$

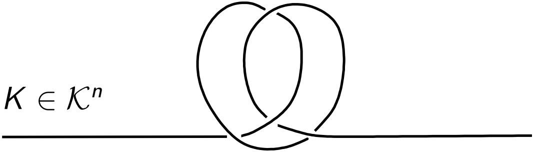
For $dim(M) + 3 \leq dim(N)$*, same is true for* π_**.*

Note that when M is 1-dimensional, N has to be at least 4-dimensional in both conditions.

Let's see how this theory applies in the case of *long knots*:

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

$$\begin{aligned}\mathcal{K}^n &= \{\text{embeddings } \; K : \mathbb{R} \hookrightarrow \mathbb{R}^n, \text{ fixed outside a compact set}\} \\ &= \textit{space of long knots}\end{aligned}$$



$K \in \mathcal{K}^n$

Classical knot theory is concerned with computing

- $H_0(\mathcal{K}^3)$, which is generated (over $\mathbb{R}$, say) by knot types, i.e. by isotopy classes of knots (*isotopy* is homotopy in the space of embeddings); and
- $H^0(\mathcal{K}^3)$, the set of knot invariants, i.e. locally constant ($\mathbb{R}$-valued) functions on $\mathcal{K}^3$, i.e. functions that take the same value on isotopic knots.

However, higher (co)homology and homotopy are also interesting, even when $n > 3$ (even though H^0 and H_0 are trivial in this case).

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

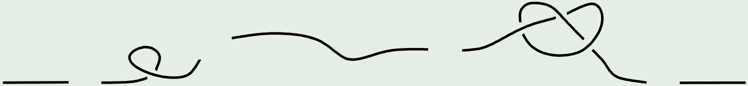
To construct $T_k\mathcal{K}^n$, let $I_1, \ldots, I_{k+1}$ be disjoint subintervals of $\mathbb{R}$ and

$$\emptyset \neq S \subseteq \{1, \ldots, k+1\}.$$

Then let

$$\mathcal{K}^n_S = \mathrm{Emb}(\mathbb{R} \setminus \bigcup_{i \in S} I_i,\ \mathbb{R}^n) = \text{space of ``punctured knots''}$$

Example



an element of $\mathcal{K}^n_{\{1,2,3,4\}}$

These spaces are not very interesting on their own, and are in fact connected even for $n = 3$. But...

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

Have restriction maps $\mathcal{K}^n_S \to \mathcal{K}^n_{S \cup \{i\}}$ given by punching another hole. These spaces and maps then form a diagram of knots with holes (such a diagram is sometimes called a *punctured cube*).

Example

When $k = 2$, we get

$$\begin{tikzcd}
 & & \mathcal{K}^n_{\{1\}} \arrow[dr] \arrow[dd] & \\
 & \mathcal{K}^n_{\{2\}} \arrow[rr] \arrow[dd] & & \mathcal{K}^n_{\{1,2\}} \arrow[dd] \\
\mathcal{K}^n_{\{3\}} \arrow[rr] \arrow[dr] & & \mathcal{K}^n_{\{1,3\}} \arrow[dr] & \\
 & \mathcal{K}^n_{\{2,3\}} \arrow[rr] & & \mathcal{K}^n_{\{1,2,3\}}
\end{tikzcd}$$

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

Definition

The kth stage of the Taylor tower for $\mathcal{K}^n$, $n \geq 3$, is the homotopy limit of the punctured cube from the previous slide. In other words,

$$T_k\mathcal{K}^n = \operatorname*{holim}_{\emptyset \neq S \subseteq \{1,..,k+1\}} \mathcal{K}^n_S.$$

Homotopy limit of a diagram should be thought of as the limit, namely the subspace of the product of the spaces in the diagram consisting of points that are compatible with the maps in the diagram, but "fattened up" so that it is made homotopy invariant.

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

Not hard to see what this homotopy limit is: The punctured cubical diagram from before can be redrawn as

$$\begin{tikzcd}
 & & \mathcal{K}^n_{\{1\}} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{K}^n_{\{1,3\}} \arrow[r] & \mathcal{K}^n_{\{1,2,3\}} & \mathcal{K}^n_{\{1,2\}} \arrow[l] & \\
\mathcal{K}^n_{\{3\}} \arrow[ur] \arrow[rr] & & \mathcal{K}^n_{\{2,3\}} \arrow[u] & & \mathcal{K}^n_{\{2\}} \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Then a point in $T_2\mathcal{K}^n$ is

- A point in each $\mathcal{K}^n_{\{i\}}$ (once-punctured knot);
- A path in each $\mathcal{K}^n_{\{i,j\}}$ (isotopy of a twice-punctured knot) ;
- A two-parameter path in $\mathcal{K}^n_{\{1,2,3\}}$ (two-parameter isotopy of a thrice-punctured knot); and
- Everything is compatible with the restriction maps.

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

There is a map

$$\mathcal{K}^n \longrightarrow T_k\mathcal{K}^n$$

given by punching holes in the knot (the isotopies in the homotopy limit are thus constant).

Easy to see: For $k \geq 3$, $\mathcal{K}^n$ is the actual pullback (limit) of the subcubical diagram.

So the strategy is to replace the limit, which is what we really care about, by the homotopy limit, which is hopefully easier to understand.

There is also a map, for all $k \geq 1$,

$$T_k\mathcal{K}^n \longrightarrow T_{k-1}\mathcal{K}^n,$$

since the diagram defining $T_{k-1}\mathcal{K}^n$ is a subdiagram of the one defining $T_k\mathcal{K}^n$ and hence the homotopy limit of the bigger diagram maps to the homotopy limit of the smaller one.

Putting these maps and spaces together, we get the Taylor tower for $\mathcal{K}^n$, $n \geq 3$:

2. Taylor tower for the space of long knots in $\mathbb{R}^n$, $n \geq 3$

$$\begin{tikzcd}
 & T_\infty \mathcal{K}^n \arrow[d] \\
 & \vdots \arrow[d] \\
\mathcal{K}^n \arrow[uur] \arrow[r] \arrow[dr] \arrow[dddr] & T_k \mathcal{K}^n \arrow[d] \\
 & T_{k-1} \mathcal{K}^n \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_0 \mathcal{K}^n
\end{tikzcd}$$

By Goodwillie-Klein-Weiss, this tower converges on homotopy and (co)homology for $n \geq 4$.

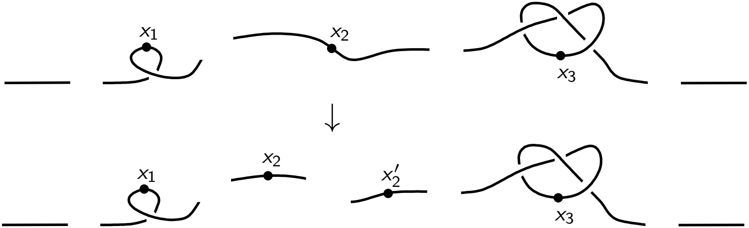
3. Cosimplicial model for the Taylor tower

$$\begin{aligned}\mathrm{Conf}(p,\mathbb{R}^n) &= \{(x_1,x_2,\dots,x_p)\in(\mathbb{R}^n)^p : x_i\neq x_j \text{ for } i\neq j\}\\ &= \textit{configuration space of } p \textit{ points in } \mathbb{R}^n\end{aligned}$$

By retracting the arcs of a punctured knot, we get

$$\mathcal{K}^n_S \simeq \mathrm{Conf}(|S|-1,\mathbb{R}^n).$$

The restriction maps “add a point”:



To make this precise, need *cosimplicial spaces*.

3. Cosimplicial model for the Taylor tower

Definition

A *cosimplicial space* $X^\bullet$ is a sequence of spaces $\{X^n\}_{n=0}^{\infty}$ with *coface* maps

$$d^i \colon X^n \longrightarrow X^{n+1}, \ \ 0 \le i \le n+1$$

and *codegeneracy* maps

$$s^i \colon X^{n+1} \longrightarrow X^n, \ \ 0 \le i \le n$$

which satisfy the relations

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, && i < j; \\ s^j d^i &= \begin{cases} d^i s^{j-1}, & i < j; \\ Id, & i < j, \ i = j+1; \\ d^{i-1} s^j, & i > j+1; \end{cases} \\ s^j s^i &= s^{i-1} s^j, && i > j. \end{aligned} \tag{1}$$

3. Cosimplicial model for the Taylor tower

Can picture $X^\bullet$ as

$$\begin{tikzcd}
X^\bullet = \Big(X^0 \arrow[r, shift left=4, "d^0"] \arrow[r, shift right=4, "d^1"] & X^1 \arrow[l, "s^0"'] \arrow[r, shift left=8, "d^0"] \arrow[r, "d^1"] \arrow[r, shift right=8, "d^2"] & X^2 \cdots \Big). \arrow[l, shift right=4, "s^0"'] \arrow[l, shift left=4, "s^1"']
\end{tikzcd}$$

Example

The *cosimplicial simplex* $\Delta^\bullet$ has the standard simplex Δ^k as its kth space, and the cofaces and codegeneracies are given by inclusions of faces and projections onto faces.

Let

$$\operatorname{Tot} X^\bullet = \operatorname{Map}(\Delta^\bullet, X^\bullet)$$

$$\operatorname{Tot}^k X^\bullet = \operatorname{Map}\big((k\text{th truncation of } \Delta^\bullet), (k\text{th truncation of } X^\bullet)\big)$$

Alternatively, can think of $\operatorname{Tot} X^\bullet$ ($\operatorname{Tot}^k X^\bullet$) as the homotopy limit of the diagram $X^\bullet$ (truncation of $X^\bullet$). Also have tower of partial totalizations

$$\operatorname{Tot} X^\bullet \longrightarrow \cdots \longrightarrow \operatorname{Tot}^k X^\bullet \longrightarrow \cdots \longrightarrow \operatorname{Tot}^0 X^\bullet$$

where $\operatorname{Tot} X^\bullet$ is the inverse limit.

3. Cosimplicial model for the Taylor tower

Cosimplicial spaces are very important in homotopy theory:

- Any diagram of spaces $\mathcal{X}$ can be turned into a cosimplicial space $X^\bullet$ in such a way that

$$\operatorname{holim} \mathcal{X} \simeq \operatorname{Tot} X^\bullet;$$

- Can associate to any $X^\bullet$ the *Bousfield-Kan (co)homology and homotopy spectral sequences* which try to compute

$$\mathrm{H}^*(\operatorname{Tot} X^\bullet), \;\; \mathrm{H}_*(\operatorname{Tot} X^\bullet), \text{ and } \pi_*(\operatorname{Tot} X^\bullet).$$

Let's get back to the Taylor tower for $\mathcal{K}^n$:

3. Cosimplicial model for the Taylor tower

Definition

Let $\mathrm{Conf}\langle k, \mathbb{R}^n\rangle$ be (a slight variation of) the *Fulton-MacPherson compactification* of $\mathrm{Conf}(k, \mathbb{R}^n)$.

Some properties:

- $\mathrm{Conf}\langle k, \mathbb{R}^n\rangle$ is homotopy equivalent to $\mathrm{Conf}(k, \mathbb{R}^n)$;
- $\mathrm{Conf}\langle k, \mathbb{R}^n\rangle$ is (almost) a manifold with corners;
- Boundary of $\mathrm{Conf}\langle k, \mathbb{R}^n\rangle$ is characterized by points colliding with directions and relative rates of collisions kept track of;
- Stratification of the boundary given by stages of collisions of points; this stratification is encoded by trees;
- Works for configurations in any manifold, not just $\mathbb{R}^n$.

3. Cosimplicial model for the Taylor tower

Definition

Let $(K^n)^\bullet$ be the cosimplicial space

$$(K^n)^\bullet = \Big(\begin{tikzcd} \mathrm{Conf}\langle 0,\mathbb{R}^n\rangle \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Conf}\langle 1,\mathbb{R}^n\rangle \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathrm{Conf}\langle 2,\mathbb{R}^n\rangle \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd} \Big),$$

where the cofaces $\overset{d^i}{\to}$ are doubling (diagonal) maps and the codegeneracies $\overset{s^i}{\leftarrow}$ are forgetting maps.

Theorem (Sinha)

For $n \geq 2$ and $k \geq 0$, $\mathrm{Tot}^k(K^n)^\bullet \simeq T_k\mathcal{K}^n$ (and hence $\mathrm{Tot}(K^n)^\bullet \simeq T_\infty\mathcal{K}^n$).

Combining this with the Godwillie-Klein-Weiss result about the convergence of the Taylor tower, we get

Corollary

For $n \geq 4$, $\mathrm{Tot}(K^n)^\bullet \simeq \mathcal{K}^n$.

3. Cosimplicial model for the Taylor tower and operads

It turns out that compactified configuration spaces form an *operad* and that $(K^n)^\bullet$, $n \geq 4$, arises from this operad (in the sense of Gerstenhaber and Voronov). Using a general theory developed by McClure and Smith, it follows that $\mathrm{Tot}(K^n)^\bullet$ admits an action of the *little discs operad* for any $n \geq 3$. A consequence is

Theorem (Sinha)

For $n \geq 4$ and for some space Y,

$$\mathcal{K}^n \simeq \Omega^2 Y.$$

Remarks:

1. Dwyer and Hess have described Y in terms of operad maps.
2. Budney has an action of the little discs operad on $\mathcal{K}^n$, $n \geq 3$, that is geometric. It would be interesting to relate that action to the one described here for $n \geq 4$. (It is not clear, however, whether our action extends to $\mathcal{K}^3$ since we do not have convergence of the Taylor tower in that case.)

4. Application to homology of $\mathcal{K}^n$, $n \geq 4$

Recall that we have the Bousfield-Kan homology spectral sequence for $(K^n)^\bullet$, $n \geq 3$. It starts with

$$E^1_{-p,q} = \mathrm{H}_q(\mathrm{Conf}(p, \mathbb{R}^n)).$$

For $n \geq 4$, this spectral sequence converges to $\mathrm{H}_*(\mathrm{Tot}(K^n)^\bullet)$, and hence to $\mathrm{H}_*(\mathcal{K}^n)$ by Goodwillie-Klein-Weiss.

Theorem (Lambrechts-Turchin-V. for $n \geq 4$, Kontsevich/V. for $n = 3$ on the diagonal, Moriya/Songhafouo-Tsopméné for $n = 3$ everywhere)

This homology spectral sequence collapses rationally at the E^2 page for $n \geq 3$.

Main ingredient in the proof: Kontsevich's rational formality of the little n-discs operad (plus model category techniques for $n = 3$).

(Collapse also true for the *homotopy* spectral sequence for $n \geq 4$; this is due to Arone-Lambrechts-Turchin-V.)

4. Application to homology of $\mathcal{K}^n$, $n \geq 4$

So for $n \geq 4$, the homology of the E^2 page is the homology of $\mathcal{K}^n$. Fancier way to say this, using the fact that $(K^\bullet)^n$ comes from the little cubes operad, is

Theorem

For $n \geq 4$, $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q}) \cong \mathrm{H}\,\mathrm{H}_*(\mathcal{POISS}_{n-1})$, where $\mathcal{POISS}_{n-1}$ is the operad obtained by taking the homology of the little n-cubes operad.

The main point:

$\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$ is built out of $\mathrm{H}_*(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{Q})$, which is understood. In fact, it can be represented combinatorially with graph complexes. So we have a nice combinatorial description of $\mathrm{H}_*(\mathcal{K}^n; \mathbb{Q})$, $n \geq 4$.

The cohomological version is

Corollary

The differential graded algebra model for $\mathrm{H}^*(\mathcal{K}^n; \mathbb{Q})$, $n \geq 4$, is

$$\left(\bigoplus_{p=0}^{\infty} s^{-p}\, \mathrm{H}^*(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{Q}), \sum \pm \mathrm{H}^*(d^i) \right).$$

5. Application to finite type invariants of $\mathcal{K}^3$

Want to see what this machinery can say about *knot invariants*, namely elements of $\mathrm{H}^0(\mathcal{K}^3)$. The cohomology spectral sequence for $(K^3)^\bullet$ is *trying* to compute these in degree zero – we do not even know that it converges to $\mathrm{H}^0(\mathrm{Tot}(K^3)^\bullet)$, let alone to $\mathrm{H}^0(\mathcal{K}^3)$. More precisely, we just have maps

$$\bigoplus_p E_2^{-p,p} \longrightarrow \mathrm{H}^0(\mathrm{Tot}(K^3)^\bullet) \longrightarrow \mathrm{H}^0(\mathcal{K}^3),$$

where $E_2^{-p,p}$ is on the diagonal of the spectral sequence, i.e. in total degree 0. We do not know if these maps are isomorphisms.

It is not hard to see that $E_2^{-p,p}$ can be represented by chord diagrams modulo *four-term relation* from the theory of *finite type knot invariants*.

Finite type knot invariants have received much attention in recent years since they are conjectured to *separate knots*, i.e. form a *complete set of knot invariants*.

5. Application to finite type invariants of $\mathcal{K}^3$

Theorem (V.)

The Taylor tower for $\mathcal{K}^3$ classifies finite type knot invariants. More precisely, for each $k \geq 0$, there is an isomorphism (over $\mathbb{R}$)

$$\mathrm{H}^0(T_{2k}\mathcal{K}^3) \cong \{\textit{finite type } k \textit{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}^3).$$

(And $\mathrm{H}^0(T_{2k}\mathcal{K}^3) \cong \mathrm{H}^0(T_{2k+1}\mathcal{K}^3)$.)

Main ingredient in the proof: Configuration space integrals.

This theorem puts finite type theory into a homotopy-theoretic setting.

6. Two-variable manifold calculus

If $M = P \coprod Q$, can apply two-variable calculus (Munson-V.) for contravariant functors F on $\mathcal{O}(P) \times \mathcal{O}(Q)$ (rather than on $\mathcal{O}(P \coprod Q)$). Get bitower

$$\begin{tikzcd}
T_{0,\infty}F(-,-) & \cdots \arrow[l] & \cdots \arrow[l] & T_{\infty,\infty}F(-,-) \arrow[l] \arrow[d] \\
\vdots \arrow[from=1-1, d] \arrow[d] & & & \vdots \arrow[d] \\
T_{0,1}F(-,-) \arrow[d] & T_{1,1}F(-,-) \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & T_{\infty,1}F(-,-) \arrow[l] \arrow[d] \\
T_{0,0}F(-,-) & T_{1,0}F(-,-) \arrow[l] & \cdots \arrow[l] & T_{\infty,0}F(-,-) \arrow[l]
\end{tikzcd}$$

- Connection to single-variable calculus: $T_k F = \operatorname*{holim}_{k_1+k_2 \leq k} T_{k_1,k_2}F$;
- Have same convergence result as in ordinary manifold calculus: For $F = \mathrm{Emb}(P \coprod Q, N)$ and same dimensional assumptions, the bitower converges.

7. Multi-cosimplicial model for links

Let $n \geq 3$ and $m \geq 1$. Define

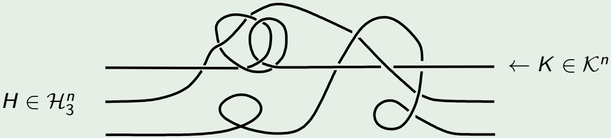
$$\begin{aligned}
\mathcal{L}^n_m &= \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of long (string) links} \\
\mathcal{H}^n_m &= \{\text{link maps} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of homotopy long (string) links} \\
\mathcal{B}^n_m &= \{\text{embeddings with positive derivative} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\
&= \textit{space of pure braids}
\end{aligned}$$

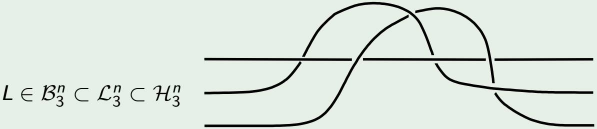
- All maps are standard outside a compact set;
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint.

7. Multi-cosimplicial model for links

- $\mathcal{B}^n_m \subset \mathcal{L}^n_m \subset \mathcal{H}^n_m$;
- In $\pi_0(\mathcal{H}^n_m)$, can pass a strand through itself so this can be thought of as space of "links without knotting".

Example





7. Multi-cosimplicial model for links

Let $I_1, \ldots, I_{k_1+1}$ be disjoint intervals in $\mathbb{R}$. Same for $J_1, \ldots, J_{k_2+1}$. Then

Definition

$$T_{k_1,k_2}\mathcal{L}_2^n = \operatorname*{holim}_{\substack{\emptyset \neq S_1 \subseteq \{1,\ldots,k_1\} \\ \emptyset \neq S_2 \subseteq \{1,\ldots,k_2\}}} \mathrm{Emb}\left((\mathbb{R} \setminus \bigcup_{i \in S_1} I_i) \coprod (\mathbb{R} \setminus \bigcup_{j \in S_j} J_j)\ ,\ \mathbb{R}^n \right).$$

Now get a bicosimplicial space $(L^n)^{\bullet,\bullet}$ whose (k_1, k_2) entry is $\mathrm{Conf}\langle k_1 + k_2, \mathbb{R}^n \rangle$ (can "double" and "forget" points in two directions).

Let $\mathrm{Tot}^{k_1,k_2}(L^n)^{\bullet,\bullet}$ be the (k_1, k_2)th partial totalization of $(L^n)^{\bullet,\bullet}$.

Proposition (Munson-V.)

For $n \geq 2$ *and* $k_1, k_2 > 0$, $\mathrm{Tot}^{k_1,k_2}(L^n)^{\bullet,\bullet} \simeq T_{k_1,k_2}\mathcal{L}_2^n$.

7. Multi-cosimplicial model for links

Want to associate a spectral sequence to this bicosimplicial space. To do this, first take the *diagonal cosimplicial space*

$$(L^n)^{\bullet,\bullet}_{diag} = \left\{ \mathrm{Conf}\langle 2k, \mathbb{R}^n \rangle \right\}_{k=0}^{\infty}.$$

It is not hard to see that

$$\mathrm{Tot}(L^n)^{\bullet,\bullet} = \mathrm{Tot}(L^n)^{\bullet,\bullet}_{diag}.$$

As before, have the Bousfield-Kan homology spectral sequence for $(L^n)^{\bullet,\bullet}_{diag}$, $n \geq 3$, with

$$E^1_{-p,q} = \mathrm{H}_q(\mathrm{Conf}(2p, \mathbb{R}^n))$$

which, for $n \geq 4$, converges to

$$\mathrm{H}_*(\mathrm{Tot}(L^n)^{\bullet,\bullet}_{diag}) = \mathrm{H}_*(\mathcal{L}^n_2).$$

(Similar for the homotopy spectral sequence.)

7. Modifications for homotopy string links and braids

Recall the spaces $\mathcal{H}_2^n$ and $\mathcal{B}_2^n$ of homotopy string links and braids. Again have bicosimplicial models for the bitowers of punctured homotopy links and braids, as well as their diagonal cosimplicial spaces, except:

- For $\mathcal{H}_2^n$, the kth space in the diagonal cosimplicial space is

  $$\mathrm{Conf}\langle k, k; \mathbb{R}^n\rangle = \text{compactification of}$$
  $$\{(x_1, \dots, x_k, y_1, \dots, y_k) \in (\mathbb{R}^n)^{2k} : x_i \neq y_j\}$$

  This is a kind of a compactified "partial configuration space" or a complement of a hyperspace arrangement.
- For $\mathcal{B}_2^n$, the kth space in the diagonal cosimplicial space is

  $$(\mathrm{Conf}\langle 2, \mathbb{R}^{n-1}\rangle)^k$$

  and this turns out to give the standard cosimplicial model for $\Omega\,\mathrm{Conf}(2, \mathbb{R}^{n-1})$ (which is exactly what braids are).

7. Generalization to more than two strands

Generalization to m-component links is straightforward:
Get, for $n \geq 3$,

- m-dimensional Taylor towers for $\mathcal{L}^n_m$, $\mathcal{H}^n_m$, and $\mathcal{B}^n_m$;
- m-cosimplicial models for these towers;
- diagonal cosimplicial spaces
  - $(L^n)^{\bullet,\bullet,\dots,\bullet}_{diag}$ consisting of $\mathrm{Conf}\langle kp, \mathbb{R}^n\rangle$, $k \geq 0$;
  - $(H^n)^{\bullet,\bullet,\dots,\bullet}_{diag}$ consisting of $\mathrm{Conf}\langle k, k, \dots, k; \mathbb{R}^n\rangle$, $k \geq 0$; and
  - $(B^n)^{\bullet,\bullet,\dots,\bullet}_{diag}$ consisting of $(\mathrm{Conf}\langle m, \mathbb{R}^{n-1}\rangle)^k$, $k \geq 0$

  modeling the towers (so their totalizations are equivalent to inverse limits of the towers);
- Bousfield-Kan homology (and homotopy) spectral sequences for these cosimplicial spaces.
- for $n \geq 4$,

  $$\mathrm{H}_*(\mathrm{Tot}(L^n)^{\bullet,\bullet,\dots,\bullet}_{diag}) \cong \mathrm{H}_*(\mathcal{L}^n_m) \quad \text{and} \quad \mathrm{H}_*(\mathrm{Tot}(B^n)^{\bullet,\bullet,\dots,\bullet}_{diag}) \cong \mathrm{H}_*(\mathcal{B}^n_m)$$

  (same for π_*)

So how many of the results we had for knots carry over?

8. Application to link cohomology

Theorem (Munson-V.)

For $n \geq 4$, the homology spectral sequences for all three cosimplicial spaces converge to their totalization. The spectral sequences for $(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ and $(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ hence converge to $\mathcal{L}^n_m$ and $\mathcal{B}^n_m$.

- Do the rational homology (and homotopy) spectral sequences for $(L^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ and $(B^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ collapse at E^2 for $n \geq 3$ (in the spirit of Lambrechts-Turchin-V./Moriya/Songhafouo-Tsopméné)?

Consequence: A combinatorial description, via graph complexes, of rational homology of $\mathcal{L}^n_m$ and $\mathcal{B}^n_m$ (latter is already understood).

- Does the rational homology (and homotopy) spectral sequence for $(H^n)^{\bullet,\bullet,\ldots,\bullet}_{diag}$ collapse at E_2 for $n \geq 3$?

This seems to be a harder problem, and solving it would *not necessarily* give a description of rational cohomology of $\mathcal{H}^n_m$ since we do not know if the Taylor tower for $\mathcal{H}^n_m$ converges.

8. Application to finite type link invariants

But if we in particular had collapse on the diagonal for $n = 3$ for the three cosimplicial spaces, we would also have

Consequence: Taylor multi-towers classify finite type invariants of $\mathcal{L}_m^3$, $\mathcal{H}_m^3$, and $\mathcal{B}_m^3$.

(The connection between finite type invariants and Taylor multi-towers for these three spaces is given by configuration space integrals as in the case of knots.)

Further consequence: Since *Milnor invariants* of $\mathcal{H}_m^3$ are finite type, this would place these classical invariants in the context of manifold calculus of functors.

Further work

So the main message is

Taylor towers contain information about topology of knots and links.

Further questions:

- Show that the Taylor tower for $\mathcal{H}^n_m$, $n \geq 4$, converges;
- Figure out what the Taylor tower for all the spaces mentioned in this talk (including knots) converge to for $n = 3$ (group completion?);
- Reprove, in the setting of Taylor towers, that finite type invariants separate braids (Kohno, Bar-Natan) and homotopy string links (Habegger-Lin);
- See if this helps in proving the same result for knots and links;
- Generalize Milnor invariants to homotopy links of spheres (or planes) in any dimension and connect with work of Koschorke. Show, using manifold calculus, that these generalizations suffice for separation of link maps of spheres.

Thank you!