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CONFIGURATION SPACE INTEGRALS AND THE COHOMOLOGY OF THE SPACE OF HOMOTOPY STRING LINKS

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ABSTRACT

Configuration space integrals have been used in recent years for studying the cohomology of spaces of (string) knots and links in $\mathbb{R}^n$ for $n > 3$ since they provide a map from a certain differential graded algebra of diagrams to the deRham complex of differential forms on the spaces of knots and links. We refine this construction so that it now applies to the space of homotopy string links — the space of smooth maps of some number of copies of $\mathbb{R}$ in $\mathbb{R}^n$ with fixed behavior outside a compact set and such that the images of the copies of $\mathbb{R}$ are disjoint — even for $n = 3$. We further study the case $n = 3$ in degree zero and show that our integrals represent a universal finite type invariant of the space of classical homotopy string links. As a consequence, we deduce that Milnor invariants of string links can be written in terms of configuration space integrals.

Keywords: Configuration space integrals; links; homotopy links; finite type invariants; chord diagrams; weight systems.

Mathematics Subject Classification 2010: Primary: 57Q45; Secondary: 57M27, 81Q30, 57R40

Contents

1. Introduction

This paper is concerned with the study of the cohomology of the space of homotopy string links (or long homotopy links) $\mathcal{H}_m^n$ using *configuration space integrals*, also known as *Bott–Taubes integrals*. This is the space of smooth maps of m copies of $\mathbb{R}$ in $\mathbb{R}^n$ where the images of the various copies of $\mathbb{R}$ are disjoint and where the map is fixed outside some compact set (see Definition 2.3). Our main results are:

(i) For $m \geq 1$ and $n \geq 4$, there exists a certain differential algebra of diagrams $\mathcal{HD}_k^d$, bigraded by two natural numbers called the *defect* d (or *degree* in the terminology of [7]) and *order* k. There exists a differential algebra map

$$I_{\mathcal{H}} : \mathcal{HD}_k^d \to \Omega^{k(n-3)+d}(\mathcal{H}_m^n), \tag{1}$$

where Ω^* stands for the deRham complex of differential forms (Theorem 4.33). Defining the main degree in $\mathcal{HD}_k^d$ to be $k(n-3)+d$ makes $\mathcal{HD}$ into a (singly graded) differential graded algebra and the above map into a map of differential graded algebras. When the defect $d = 0$, the induced map in cohomology is injective.

(ii) For $m \geq 1$ and $n = 3$, we get a similar map for defect $d = 0$ (which coincides with main degree zero when $n = 3$):

$$I_{\mathcal{H}} : \mathcal{HD}_k^0 \to \Omega^0(\mathcal{H}_m^n), \tag{2}$$

which takes closed forms to closed forms and is injective in cohomology. This map produces all finite type invariants of homotopy string links (Theorem 5.8).

(iii) As a consequence of the previous result, we can express Milnor invariants of homotopy string links in $\mathbb{R}^3$ (Theorem 5.13) completely in terms of configuration space integrals. Using the weight systems for these invariants, we can explicitly write down these formulae up to lower-order finite type invariants (which themselves can be expressed as configuration space integrals).

The first two results parallel those for string (i.e. long) knots $\mathcal{K}^n$, i.e. embeddings of $\mathbb{R}$ in $\mathbb{R}^n$ [5, 7, 8, 29]. More generally, they parallel results for string links $\mathcal{L}_m^n$, i.e. embeddings of m copies of $\mathbb{R}$ in $\mathbb{R}^n$, where all maps are always prescribed outside some compact set. In the process of obtaining our results, we provide an erratum to [31], which considered the case of string links. At the same time, these results are also very different from the case of string knots/links. To explain, we first briefly review the standard construction of the map

$$\overline{I}_{\mathcal{L}} : \mathcal{LD} \to \Omega^*(\mathcal{L}_m^n) \tag{3}$$

corresponding to that in (1) and is familiar from the literature [7, 31]. In particular, $\mathcal{LD}$ is a familiar diagram complex associated to the space of string links.

To produce forms on $\mathcal{L}_m^n$, one first creates fiber bundles of configuration spaces over this space. Each bundle depends on a diagram in $\mathcal{LD}$. A diagram has vertices that abstractly represent configurations of points on and off a link, and its edges prescribe a way to pull back copies of the volume $(n-1)$-form from the sphere S^{n-1}

to the total space of the bundle. We then integrate this pullback form along the fiber, thereby producing a form on $\mathcal{L}^n_m$. One of the main reasons this construction works is that ordinary embedded links behave well with respect to restriction, i.e. the restriction map for links is a fibration by the Isotopy Extension Theorem.

The situation is different for $\mathcal{H}^n_m$ because homotopy links are not embeddings and the restriction map is far from a fibration (see Sec. 4.2.2). Thus, the obvious generalization of the above fails to extend to $\mathcal{H}^n_m$. The main contribution of this paper is a refinement of the construction of the fiber bundles which makes it possible to integrate over $\mathcal{H}^n_m$. The short explanation of this refinement is that, in the construction of $\overline{I}_{\mathcal{L}}$, only vertices of the diagram determine the bundle, while in our construction, both vertices and edges are relevant. This leads to breaking up the diagram according to its "grafts" (see Definitions 4.10 and 4.13) and the construction of what is essentially a product bundle over the set of graft components. In this fashion we construct a new map

$$I_{\mathcal{L}} : \mathcal{LD} \to \Omega^*(\mathcal{L}^n_m), \tag{4}$$

identify a subcomplex $\mathcal{HD} \subset \mathcal{LD}$, exhibit the map from Eq. (1), and show that the diagram

$$\begin{tikzcd} \mathcal{HD} \arrow[r, hook] \arrow[d, "I_{\mathcal{H}}"'] & \mathcal{LD} \arrow[d, "I_{\mathcal{L}}"] \\ \Omega^*(\mathcal{H}^n_m) \arrow[r] & \Omega^*(\mathcal{L}^n_m) \end{tikzcd}$$

commutes. After we define $I_{\mathcal{L}}$ and show how it restricts to the map $I_{\mathcal{H}}$, we show in Proposition 4.24 that the old integration map $\overline{I}_{\mathcal{L}}$ and our map $I_{\mathcal{L}}$ produce the same form. Thus, our construction is indeed a refinement of the one considered by others.

One interesting attribute of our construction of $I_{\mathcal{H}}$ is that this map can be defined even when $n = 3$, which is not the case with $I_{\mathcal{L}}$. The reason is that the issue of the vanishing of the integration along a certain part of the boundary of the bundle, the so-called *anomalous faces*, is not present for homotopy links (see Remark 5.1). This is potentially an exciting feature since it means that the map $I_{\mathcal{H}}$ might contain interesting information about the topology of the space of classical homotopy string links.

The anomalous face also makes an appearance in the study of finite type invariants of knots and links via configuration space integrals [29–31]. As stated in (ii) above, we extend this study to the case of homotopy string links. The difference is that, for (string) knots and links, these integrals represent a universal finite type invariant only up to an indeterminacy due to the non-vanishing of anomalous faces (see Sec. 5.1). However, this is not a problem for homotopy string links and in Theorem 5.8 we give the correspondence between weight systems (functionals on diagrams with defect zero satisfying some relations) and finite type invariants of homotopy string links without any indeterminacy.

Theorem 5.8 connects to other work that has been done on finite type invariants of homotopy string links. To show that $I_{\mathcal{H}}$ represents the universal finite type

invariant of homotopy string links, we first show that the zeroth cohomology of the complex $\mathcal{HD}$ gives a certain vector space of diagrams that has already been studied [4, 18, 20, 31]. Our construction, however, is dictated by geometry — we have arrived at $\mathcal{HD}$ by looking for spaces we could integrate over to get forms on $\mathcal{H}_m^n$. Further, we are concerned with all $n \geq 3$, and for $n = 3$ and degree zero (which is also defect zero) we happen to have obtained the "correct" diagrams and relations. This means that our approach is indeed a generalization, with a new perspective, of existing work.

Since Milnor invariants of homotopy string links are known to be finite type, Theorem 5.8 immediately gives a novel construction for Milnor invariants entirely in terms of configuration space integrals (as mentioned in (iii)). Further, some connections between tree diagrams and Milnor invariants arise naturally from our construction, and this will be pursued in future work. More details about the planned work on Milnor invariants are given in Sec. 5.4.

The philosophy in this paper is thus to reconstruct all the ingredients of the map (4), but in an improved and refined fashion, and then show at every important instance of the construction how everything works when one restricts to the case of homotopy string links. Consequently, we have had to be precise and detailed about the definition and structures in the diagram complex $\mathcal{LD}$, the fiber bundles mentioned earlier, the defect zero case, etc. This has required us to fill in some of the details that have been missing from the literature. Some instances of this are:

- the graph complex $\mathcal{LD}$ is now defined purely combinatorially (it had largely been done through pictures before, and mainly for the case of knots);
- the correspondence between the shuffle product on $\mathcal{LD}$ and the wedge product on $\Omega^*(\mathcal{L}_m^n)$ is elucidated;
- the *STU* and *IHX* relations in defect zero are derived from the graph complex;
- essentially all the details of the proof that configuration space integrals represent a universal finite type invariant of embedded string links and homotopy string links are given (the most complete proof for knots is in [30]; in particular, our work provides an erratum to [31], which treated the string links case);

In addition, the work here unifies and extends many seemingly disparate results in the subject of configuration space integrals (the case $n > 3$ is in literature usually treated separately from the case $n = 3$). All of this makes for a self-contained and thorough treatment of how configuration space integrals are used in knot and link theory. We hope that in addition to establishing some new and useful results, this paper will serve as a practical and a beneficial introduction to the subject.

Finally, it is worth noting where the results from this paper fit into the larger program of studying homotopy string links (and embedded string links) in the context of *manifold calculus of functors*. To that end, the second and the third author have developed its multivariable version [24], as well a cosimplicial model for the functor calculus Taylor tower for homotopy string links [25]. Using this

model, the plan is to show that the map $I_{\mathcal{L}}$ factors through the Taylor tower and that this tower classifies finite type invariants. This can hopefully be used to reprove the Habegger–Lin classification of homotopy links [11] as well as to extend some of their results to ordinary links. Along the way, the authors plan to study Milnor invariants in the context of manifold calculus, which continues the exploration of the connection between configuration space integrals and Milnor invariants, as well as [23], which connects manifold calculus of certain generalizations of homotopy links to generalizations of Milnor invariants.

1.1. *Organization of the paper*

- In Sec. 2, we define the spaces of string links and homotopy string links, make some observations about them, and set some notation and conventions.
- In Sec. 3, we define the diagram complex $\mathcal{LD}$ and its subcomplex $\mathcal{HD}$. Section 3.1 contains the detailed definition of $\mathcal{LD}$ and Sec. 3.2 discusses the differential and the shuffle product on this graded vector space. The subcomplex $\mathcal{HD}$ is identified in Sec. 3.3. In Sec. 3.4, we show that $\mathcal{LD}^0$ and $\mathcal{HD}^0$ consist of trivalent diagrams modulo *STU* and *IHX* relations, plus an extra relation for $\mathcal{HD}^0$ (Proposition 3.33). In that section we also describe the correspondence between trivalent and chord diagrams (Theorem 3.38). Many examples are given throughout.
- In Sec. 4, we construct the map $I_{\mathcal{L}}$ in several steps. After reminding the reader about compactifications of configuration spaces in Sec. 4.1, we first recall in Sec. 4.2.1 the standard way of building a bundle of compactified configuration spaces over the space of string links from a diagram $\Gamma \in \mathcal{LD}$. In Sec. 4.2.2, we show why this procedure fails to give bundles over the space of homotopy string links. Guided by how this procedure fails, we then go back to the complex $\mathcal{LD}$, define the graft components of a diagram in Sec. 4.2.3, and rework the definition of the bundle of configuration spaces based on these components in Sec. 4.2.4. The upshot is that these new bundles can now be defined over the space of links for any $\Gamma \in \mathcal{LD}$ or over the space of homotopy links for any $\Gamma \in \mathcal{HD} \subset \mathcal{LD}$. In Sec. 4.3, we return to the main goal — producing forms on the space of (homotopy) links — and show how the edges of a diagram give a prescription for pulling back the product of volume forms to our bundles. Finally, in Sec. 4.4, we describe how this pullback form can be pushed forward along the fiber of the bundle to $\mathcal{L}^n_m$ or $\mathcal{H}^n_m$ and give some examples. Proposition 4.24 states that the forms obtained using the standard definition of the bundles over $\mathcal{L}^n_m$ and using our refined one are the same. This allows us to unify the old configuration space integral approach for string links with a new one for homotopy string links. Our Theorem 4.33 in Sec. 4.5 shows that this integration is compatible with all the structure on $\mathcal{LD}$.
- In Sec. 5, we study the case of classical homotopy string links ($n = 3$) and prove that configuration space integrals represent a universal finite type invariant for this space (Theorem 5.8). We begin by discussing in Sec. 5.1 the anomalous

face mentioned above and then review finite type theory and its connection to the combinatorics of chord diagrams in Sec. 5.2. Section 5.3 is finally devoted to the proof of Theorem 5.8. In Sec. 5.4, we deduce some quick consequences of Theorem 5.8 in regard to Milnor invariants. That section is meant to set the stage for the further study of Milnor invariants using configuration space integrals.

2. Spaces of String Links and Homotopy String Links

In this section, we define the spaces of string links and homotopy string links and set some conventions.

2.1. *Definitions and basic facts*

Definition 2.1. Let $m \geq 1$, $n \geq 2$ be integers. Let $\sqcup_m \mathbb{R}$ denote the disjoint union of m copies of the real line. Let $\mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ denote the space of smooth maps $\sqcup_m \mathbb{R} \to \mathbb{R}^n$ which outside some compact subset of $\sqcup_m \mathbb{R}$ agree with the map which on the ith copy of $\mathbb{R}$ is given by

$$t \mapsto \left(t, |t| \left(\frac{m+1}{2} - i\right), 0, 0, \ldots, 0\right).$$

This space is endowed with the $\mathcal{C}^\infty$ topology.

The following is clear.

Proposition 2.2. $\mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ *is contractible.*

Definition 2.3.

- Let $\mathcal{L}_m^n \subset \mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ denote the space of *string* (*or long*) *links in* $\mathbb{R}^n$ *with* m *strands*. It consists of those maps $L \in \mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ which are smooth embeddings (one-to-one maps whose derivatives are of maximal rank everywhere). A path in this space is called an *isotopy*.
- Let $\mathcal{H}_m^n \subset \mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ denote the space of *string* (*or long*) *homotopy links in* $\mathbb{R}^n$ *with* m *strands*. It consists of those maps $H \in \mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ such that if x and y are points in distinct copies of $\mathbb{R}$, then $H(x) \neq H(y)$. A path in this space is called a *link-homotopy*.

Note that $\mathcal{H}_m^n$ is an example of a space of *link maps*, studied by the second author in [10, 22, 23] from the perspective of the manifold calculus of functors; one motivation for the current work was to continue this thread of inquiry.

Throughout the paper, we will often drop the adjectives "string" and "long", and refer to these objects as "links" and "homotopy links". Each link and homotopy link is oriented in the sense that all copies of $\mathbb{R}$ are given the usual orientation. The images of the copies of $\mathbb{R}$ will be called *strands*.

In the literature, a more common picture for string links is to take the ith copy of $\mathbb{R}$ to $t \mapsto (t, i, 0, 0, \ldots, 0)$ outside the fixed compact set. There is a clear

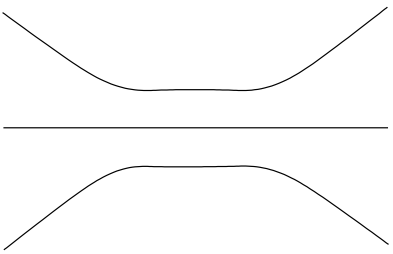

Fig. 1. The string unlink with three strands.

correspondence between these string links and string links under our definition; in particular, we think of the unlink as in Fig. 1. We have chosen our definition for technical reasons related to defining configuration space integrals for string links. This technicality is also related to an error in [31] which we will correct.

The following corollary is immediate from Proposition 2.2.

Corollary 2.4. *$\mathcal{H}_1^n$ is contractible.*

In Sec. 5, we will be interested in $\mathrm{H}^0(\mathcal{H}_m^3)$, i.e. the space of real-valued invariants of m-strand homotopy links in $\mathbb{R}^3$, so we discuss deRham forms on link spaces below. First, we make an observation which will be useful in Sec. 5.

Remark 2.5. By general position, every homotopy link is link-homotopic to an embedded link. Moreover, by the remark following [11, Definition 1.5], we can approximate a link-homotopy between embedded links by one which consists of isotopies and "crossing changes" of a strand with itself. A crossing change is a homotopy which takes place in the interior of a ball containing only two segments of a single strand, and the two segments cross during the homotopy. To check that something is an invariant of $\mathcal{H}_m^3$, it thus suffices to check that it is an invariant of $\mathcal{L}_m^3$ and that it remains unchanged under such crossing changes. This observation will be used in the proof of Proposition 5.10 and will also allow us to connect the main results of Sec. 5 to Milnor invariants since these are in fact invariants of embedded links that are also invariant under such crossing changes.

2.2. *Smooth structure and differential forms*

In this subsection, we give a brief sketch of the smooth structure and differential forms on spaces of links. The space $\mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$ can be given the structure of a smooth paracompact infinite-dimensional manifold (see [6, Sec. 3.1] as well as [27]; strictly speaking, these references treat the case of maps of S^1, but the local picture is the same in our case). Both $\mathcal{L}_m^n$ and $\mathcal{H}_m^n$ are open subsets of $\mathrm{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$, as the latter space has the $\mathcal{C}^\infty$ topology. Using similar ideas to the ones in the references mentioned, one can (with some effort) give each of these spaces the structure of a paracompact smooth infinite-dimensional manifold.

Another useful perspective on the smooth structure is via a *diffeology* on a space X, which consists of a collection of maps to X from open subsets $U \subset \mathbb{R}^k$, $k \geq 0$, called *plots*, which must satisfy certain conditions. When X is a smooth manifold, one can take the plots to be precisely all the smooth maps into X [13]. We are interested in the case where X is the infinite-dimensional manifold of smooth maps from a compact manifold K to a manifold M. In this case, this diffeology coincides with the diffeology where a plot $U \to X$ is precisely a smooth map $U \times K \to M$ (see [32, Lemma A.1.7]). In particular, for $X = \mathcal{L}^n_m$, a plot $U \to \mathcal{L}^n_m$ is a smooth map $U \times (\sqcup_m \mathbb{R}) \to \mathbb{R}^n$ such that each slice $\{u\} \times (\sqcup_m \mathbb{R}) \to \mathbb{R}^n$ is a string link. A diffeology on $\mathcal{H}^n_m$ can be defined similarly.

For any manifold M, we let $\Omega^*(M)$ denote the deRham cochain complex of differential forms on M. It is a differential graded algebra where the algebra structure is given by the wedge product of forms. The ground ring for all cohomology groups will be $\mathbb{R}$. This complex can be defined even when M is an infinite-dimensional manifold. For example, one can consider forms on open subsets of the topological vector space on which M is locally modeled and then impose the sheaf condition to construct forms on all of M. Under certain conditions on M (such as paracompactness), which are satisfied by loop spaces and hence by $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$, the cohomology of this complex computes the cohomology of M. See [6, Sec. 1.4] for details.

Using the perspective of diffeology, we could equivalently define forms on M using the open sets $U \subset \mathbb{R}^k$ mapping into M. Specifically, a form ω is an assignment of a form ω_ψ on U to each $\psi : U \to M$ such that the assignment to a plot arising from a smooth map $h : V \to U$ is $h^*\omega_\psi$. A form ω in the sense of [6] mentioned above gives rise to a form in diffeology by taking $\omega_\psi = \psi^*\omega$. Conversely, one can reconstruct a form on an infinite-dimensional manifold from its behavior on every finite-dimensional open set mapped into it. At some point it will be convenient to think of forms on $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$ in this way, namely as determined by their behaviors on finite-dimensional manifolds mapped into them.

3. Diagram Complexes for the Spaces of String Links and Homotopy String Links

In this section, we construct a diagram complex $\mathcal{LD}$ and a subcomplex $\mathcal{HD}$ that will serve as combinatorial prescriptions for producing cohomology classes on spaces of links and homotopy links. The complex $\mathcal{LD}$ has been considered before [7, 31], but the definition of $\mathcal{HD}$ appears to be new. As mentioned in Sec. 1, in this section we also fill in some details in the definition and properties of $\mathcal{LD}$.

3.1. *Diagram complex for the space of string links*

While reading this section, the reader is encouraged to refer to Fig. 2.

For a set S, we let $\mathrm{SP}_2(S)$ be the 2-fold symmetric product,

$$\mathrm{SP}_2(S) = (S \times S)/\Sigma_2,$$

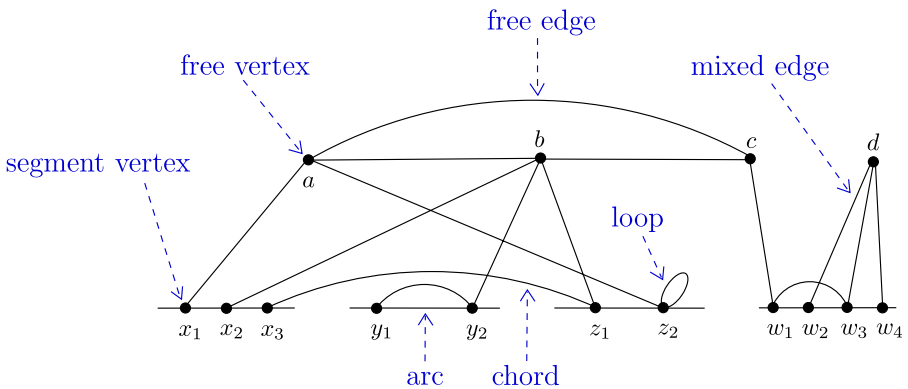

Fig. 2. A diagram with four segments. Its edges may be labeled or oriented. Each vertex is at least trivalent (the valence of, say, vertex z_2, is five).

where Σ_2 acts on the product $S \times S$ by permuting the coordinates. We think of $\mathrm{SP}_2(S)$ as the set of nonempty subsets of S of cardinality at most two. We denote points in $\mathrm{SP}_2(S)$ as sets $\{s_1, s_2\}$ where $s_1, s_2 \in S$, with the understanding that the cardinality of this set is one when $s_1 = s_2$.

Definition 3.1. A *diagram* Γ is a triple

$$\Gamma = (V(\Gamma), E(\Gamma), b_\Gamma),$$

where

- $V(\Gamma)$ is a finite ordered set called the *vertices* of Γ;
- $E(\Gamma)$ is a finite set called the *edges* of Γ; and
- $b_\Gamma : E(\Gamma) \to \mathrm{SP}_2(V(\Gamma))$ is a map.

For an edge $e \in E(\Gamma)$ with $b(e) = \{v, w\}$, we say that e *joins* v *with* w. When it is clear which diagram Γ we are speaking of we will write (V, E, b) in place of $(V(\Gamma), E(\Gamma), b_\Gamma)$.

The particular diagrams we study have a significant amount of extra structure. As we do not wish to impose cumbersome notation on the reader, we will continue to denote a diagram Γ with extra structure as a triple (V, E, b), despite the possible ambiguity. Before describing the extra structures, we need some definitions and terminology.

Definition 3.2. For a diagram $\Gamma = (V, E, b)$ and an edge $e \in E$, an *orientation* of e is a choice of injective map $b(e) \to \{-1, 1\}$.

Note that for an edge e such that $b(e)$ consists of a single vertex, there are still two possible orientations, just as there are in the case where $b(e)$ consists of two distinct vertices.

Definition 3.3. Let v, w be vertices in a diagram $\Gamma = (V, E, b)$. A *path* between v and w is a sequence $\{e_i\}_{i=1}^k$ of edges e_i such that $v \in b(e_1), w \in b(e_k)$ and $b(e_i) \cap b(e_{i+1}) \neq \emptyset$ for all i. The *length* of a path $\{e_i\}_{i=1}^k$ is equal to k, the number of elements in the sequence of edges.

Thus, the orientations of edges, if they are present, are ignored for the purposes of defining a path.

One other definition we will have to use for later is that of a connected component.

Definition 3.4. Let v be a vertex in a diagram $\Gamma = (V, E, b)$. The *connected component of* Γ *containing* v is the subdiagram (V', E', b'), where V' is the set of all vertices w that can be connected by a path to v, E' is the set of all edges that can appear in such paths, and b' is the restriction of b. A diagram is called *connected* if it has a single connected component.

Fix integers $m \geq 1$, $n \geq 3$, and let $I_1, \ldots, I_m$ be copies of the unit interval, each of which we will call a *segment* for short. The space $\sqcup_i I_i$ is an ordered set according to the natural ordering of $\{1, \ldots, m\}$ and the natural ordering of I. Thus, for $x, y \in \sqcup_i I_i$, $x \leq y$ whenever $x \in I_i$ and $y \in I_j$ and $i < j$, and when $i = j$, $x \leq y$ if this inequality holds under the usual ordering of $I = [0, 1]$.

Definition 3.5. Given integers $m \geq 1$, $n \geq 3$ as above, a *link diagram* is a diagram $\Gamma = (V, E, b)$ together with the following extra structure. For the set V of vertices, we have:

- A decomposition
$$V = V_{\text{seg}} \sqcup V_{\text{free}}$$
into ordered (possibly empty) sets, the elements of which are called *segment* and *free* vertices, respectively. In addition, we require that the induced ordering of $(V_{\text{seg}}, V_{\text{free}})$ as an ordered pair of ordered sets agrees with the ordering of V.
- A decomposition of
$$V_{\text{seg}} = V_{\text{seg},1} \sqcup \cdots \sqcup V_{\text{seg},m}$$
into disjoint sets determined by the equivalence class of an injective function $\text{seg} : V_{\text{seg}} \to \sqcup_i (I_i - \partial I_i)$ where $V_{\text{seg},k} = \text{seg}^{-1}(I_k - \partial I_k)$, and which gives rise to the ordering of V_{seg} according to the ordering of $\sqcup_i I_i$ described above. Two such injections s, s' are equivalent if they give rise to the same decomposition of V_{seg} and the same ordering on each of the sets in this decomposition according to the natural ordering of $\sqcup_i I_i$.

For the set E of edges, we have a decomposition
$$E = E_{\text{chord}} \sqcup E_{\text{mixed}} \sqcup E_{\text{free}} \sqcup E_{\text{loop}}$$

into

- *chords*, joining distinct segment vertices;
- *mixed edges*, joining a free vertex with a segment vertex;
- *free edges*, joining distinct free vertices; and
- *loops*, joining a segment vertex with itself,

respectively. Moreover, each free vertex must have a path to a segment vertex.

The *valence* of a vertex v is defined as follows. If v is a free vertex, its valence is the number of edges joining v to another vertex. If v is a segment vertex, it is the number of edges joining v to a vertex other than itself, plus twice the number of loops joining v to itself, plus two. The valence of each vertex in a link diagram is required to be at least three. In addition:

- If n is even, the set E of edges is ordered.
- If n is odd, each edge $e \in E$ is oriented.

Remark 3.6. Another terminology for segment and free vertices is "external" and "internal", respectively. This is because, in the case of knots, one has diagrams consisting of only one segment and if one is working with closed knots rather than long ones, the segment is drawn as a circle and free vertices are drawn inside it — hence "internal". The vertices on the circle are then "external". We decided that this terminology is misleading for our situation and prefer to call the vertices "segment" (those represented as lying on the m segments) and "free" (those not lying on the segments; these abstractly correspond to configuration points that are free to move in $\mathbb{R}^n$).

We will also distinguish "arcs" of a link diagram, which will be important when we define the differential, and should explain our seemingly strange definition of the valence of a segment vertex, as arcs contribute to the valence without counting as edges themselves.

Definition 3.7. For a link diagram Γ, an *arc* of Γ is a pair (v_1, v_2) of distinct segment vertices with $v_1 < v_2$ whose images under the injection $\mathrm{seg} : V_{\mathrm{seg}} \to \sqcup_i (I_i - \partial I_i)$ lie in the same segment, and such that the image of no other segment vertex lies between them.

We assume all possible arcs are present in any link diagram. Although arcs are not edges, it is useful to treat them as such at times, and so for an arc $a = (v_1, v_2)$ we define $b(a) = \{v_1, v_2\}$.

We pictorially represent a diagram in the plane with the intervals drawn as horizontal line segments, appearing in order from left to right and oriented from left to right, and each vertex as a point and each edge as an arc between vertices.

Segment vertices are drawn on the intervals, and we think of arcs as segments in the intervals which lie between adjacent segment vertices. See Fig. 2.

Definition 3.8. Link diagrams $\Gamma = (V(\Gamma), E(\Gamma), b_\Gamma)$ and $\Gamma = (V(\Gamma'), E(\Gamma'), b_{\Gamma'})$ are *isomorphic* if there are order-preserving bijections $\phi_V : V(\Gamma) \to V(\Gamma')$ and $\phi_E : E(\Gamma) \to E(\Gamma')$ respecting the decomposition of the vertex set such that if $\phi_V^* : \mathrm{SP}_2(V(\Gamma)) \to \mathrm{SP}_2(V(\Gamma'))$ denotes the induced map, then the diagram

$$\begin{tikzcd} E(\Gamma) \arrow[r, "b_\Gamma"] \arrow[d, "\phi_E"'] & \mathrm{SP}_2(V(\Gamma)) \arrow[d, "\phi_V^*"] \\ E(\Gamma') \arrow[r, "b_{\Gamma'}"] & \mathrm{SP}_2(V(\Gamma')) \end{tikzcd}$$

commutes. In addition, if n is odd (so that each edge is oriented), for each edge $e \in \Gamma$, the injections $o_{\Gamma,e} : b_\Gamma(e) \to \{-1, 1\}$ and $o_{\Gamma',\phi_E(e)} : b_{\Gamma'}(\phi(e)) \to \{-1, 1\}$ must satisfy $o_{\Gamma',\phi_E(e)} \circ \phi_V^* = o_{\Gamma,e}$. In this case, we say the pair (ϕ_V, ϕ_E) is an *isomorphism*.

If a pair (ϕ_V, ϕ_E) is simply bijections (i.e. not necessarily order-preserving) satisfying all of the subsequent properties in addition to ϕ_V being order-preserving on the segment vertices, then we say that the pair (ϕ_V, ϕ_E) is an *isomorphism of unlabeled diagrams*.

Note that "order-preserving" for edges is only relevant when the edge set is ordered.

Definition 3.9. Define *defect* of a link diagram $\Gamma = (V, E, b)$ to be

$$\mathrm{def}(\Gamma) = 2|E| - 3|V_{\mathrm{free}}| - |V_{\mathrm{seg}}|. \tag{5}$$

This is the first of the two gradings we will have in our bigraded complex.

Notice that, because we require the valence of each vertex in a link diagram to be at least three, the defect is nonnegative (there are no free vertices whose valence is less than three and there are no segment vertices whose valence is zero; if it were otherwise, the defect could be made arbitrarily negative). Thus, we can think of the defect as a measure of the failure of Γ to be trivalent. Indeed, when $\mathrm{def}(\Gamma) = 0$, the segment and free vertices are precisely trivalent (in particular, Γ cannot contain loops). We will revisit such diagrams in Sec. 3.4.

Remark 3.10. The reader acquainted with the work in [7] knows that Cattaneo, Cotta-Ramusino, and Longoni call this number the *degree* of a diagram. We use the term "defect" instead, to avoid confusion with the degree of the differential form on the space of links which a diagram gives rise to via the integration map. Then, as suggested by the referee, we can take this latter degree to be the main degree (Definition 3.15) in our bigraded complex, which will make the integration map a map of differential *graded* algebras.

Definition 3.11. When n is even (respectively, odd), define $\mathcal{LD}^d_{\text{even}}$ (respectively, $\mathcal{LD}^d_{\text{odd}}$) to be real vector spaces generated by isomorphism classes of link diagrams Γ of defect d modulo subspaces generated by the relations:

(i) If Γ contains more than one edge joining two vertices, then $\Gamma = 0$;

(ii) If n is odd and Γ and Γ' are link diagrams such that a permutation of the vertices of Γ' results in a link diagram isomorphic to Γ, then

$$\Gamma = (-1)^\sigma \Gamma',$$

where

$$\begin{aligned}\sigma = {} & (\text{order of the permutation vertices}) \\ & + (\text{number of edges with different orientation});\end{aligned}$$

(iii) If n is even and Γ and Γ' are link diagrams such that a permutation of the vertex and edge sets of Γ' results in a link diagram isomorphic to Γ, then

$$\Gamma = (-1)^\sigma \Gamma',$$

where

$$\begin{aligned}\sigma = {} & (\text{order of the permutation of segment vertices}) \\ & + (\text{order of the permutation of the edges}).\end{aligned}$$

Finally, define the graded vector spaces

$$\mathcal{LD}_{\text{even}} = \bigoplus_d \mathcal{LD}^d_{\text{even}} \quad \text{and} \quad \mathcal{LD}_{\text{odd}} = \bigoplus_d \mathcal{LD}^d_{\text{odd}}.$$

When there is no danger of confusion, i.e. when n is understood, we will refer to both $\mathcal{LD}^d_{\text{even}}$ and $\mathcal{LD}^d_{\text{odd}}$ as $\mathcal{LD}^d$ and to both $\mathcal{LD}_{\text{even}}$ and $\mathcal{LD}_{\text{odd}}$ as $\mathcal{LD}$.

Remark 3.12. The reader might argue that we should simply disallow multiple edges between a given pair of vertices rather than mod out by the subspace of such diagrams, but the differential, defined below, can introduce such edges.

Remark 3.13. Even in a fixed degree and after all the relations are imposed, $\mathcal{LD}^d$ is still an infinite-dimensional vector space: Consider, for example, the diagram consisting of three segments with one segment vertex on each segment, and a single free vertex with three edges which join it to the segment vertices. This is a diagram of degree zero. Overlaying copies of this diagram (that is, introducing new segment and free vertices and edges in a similar fashion) gives an infinite list of degree zero diagrams which are clearly independent in the vector space structure.

Definition 3.14. Define the *order* of a diagram Γ to be

$$\text{ord}(\Gamma) = |E(\Gamma)| - |V(\Gamma)_{\text{free}}|.$$

This will be the second grading in our bigraded complex.

Thus, for each $d = \mathrm{def}(\Gamma)$ and each $k = \mathrm{ord}(\Gamma)$, we have subspaces $\mathcal{LD}_k^d \subset \mathcal{LD}^d$ and $\mathcal{HD}_k^d \subset \mathcal{HD}^d$.

Note that in the case of defect zero, we also necessarily have

$$\mathrm{ord}(\Gamma) = \frac{1}{2}(|V(\Gamma)_{\mathrm{seg}}| + |V(\Gamma)_{\mathrm{free}}|).$$

In general, a diagram Γ of defect d and order k satisfies $|V(\Gamma)_{\mathrm{seg}}| + |V(\Gamma)_{\mathrm{free}}| = 2k - d$ and $|E(\Gamma)| = |V(\Gamma)_{\mathrm{free}}| + k$. Hence the number of vertices of such a diagram is fixed, and the number of edges is bounded. This means that $\mathcal{LD}_k^d$ and $\mathcal{HD}_k^d$ are finite-dimensional for any d, k.

The following definition of degree of a diagram was suggested by the referee and motivated by the fact it coincides with the degree of the differential form on the link space resulting from applying the integration map to the diagram.

Definition 3.15. Define the (*main*) *degree* of a diagram Γ to be

$$|\Gamma| = (n-1)|E(\Gamma)| - n|V(\Gamma)_{\mathrm{free}}| - |V(\Gamma)_{\mathrm{seg}}|.$$

Equivalently, for a diagram Γ of defect d and order k,

$$|\Gamma| = k(n-3) + d.$$

We will use this definition to make (singly) graded complexes out of the bigraded complexes $\mathcal{LD}_*^*$ and $\mathcal{HD}_*^*$.

Note that for $n = 3$ the degree coincides exactly with the defect.

3.2. *Algebraic structures on the diagram complex*

We now discuss the differential and the product on the space of diagrams which will make it into a differential graded algebra.

3.2.1. *The differential*

The differential of a diagram will be a signed sum of diagrams obtained from the original by "contracting" certain edges or arcs. We begin with some terminology and conventions.

Definition 3.16. Let S be a nonempty set, and let $s, t \in S$. Define

$$R_{t\to s} : \mathrm{SP}_2(S) \to \mathrm{SP}_2(S)$$

by

$$R_{t\to s}(T) = \begin{cases} T & \text{if } t \notin T; \\ (T - \{t\}) \cup \{s\} & \text{if } t \in T. \end{cases}$$

Thus, the map $R_{t\to s}$ replaces t with s. Let $\Gamma = (V, E, b)$ be a link diagram and e be a mixed or free edge of Γ, or one of its arcs, and suppose $b(e) = \{v, w\}$, where $v < w$ in the ordering of the vertices. In case e is an arc, we suppose it is represented by the pair (v, w). Note that e necessarily joins distinct vertices.

Definition 3.17. With Γ and e as above, define $\Gamma/e = (V', E', b')$ to be the link diagram such that:

- $V' = V - \{w\}$ with the induced ordering of vertices,
- $E' = E - \{e\}$ with the induced ordering/orientation of edges (if applicable), and
- $b' = R_{w\to v} \circ b$, restricted to E'.

We often refer to Γ/e as the diagram Γ with the edge/arc e contracted. The function $R_{w\to v}$ above simply replaces an edge joining w with a vertex u with the edge which joins v to u instead. This can create a loop in the case of a chord between adjacent segment vertices when the arc between them is contracted. Note that the degree is increased by contraction of a mixed/free edge or arc: if Γ has degree d, then Γ/e has degree $d + 1$. The differential is a signed sum of diagrams made from Γ by contracting all possible edges and arcs. We will use the "position" function to help keep track of these signs.

Definition 3.18. Suppose S is a finite ordered set. Define the *position function* to be the unique order-preserving bijection

$$\mathrm{pos} : S \to \{1, 2, \ldots, |S|\}.$$

When $x \in S$, we write $\mathrm{pos}(x)$ for the value of this function at $x \in S$, or $\mathrm{pos}(x : S)$ when we wish to emphasize the underlying ordered set S.

Definition 3.19. The differential

$$\delta : \mathcal{LD}_k^d \to \mathcal{LD}_k^{d+1} \tag{6}$$

is the unique linear extension to $\mathcal{LD}_k^d$ of the map defined on a diagram Γ by

$$\delta(\Gamma) = \sum_{\text{free edges, mixed edges, and arcs } e \text{ of } \Gamma} \epsilon(e)\Gamma/e. \tag{7}$$

The number $\epsilon(e)$ is equal to ± 1 depending on the parity of n and on the orderings of vertices and edges in the following way: Suppose the free/mixed edge or arc e connects vertices v and w.

- If n is odd and e is an edge or an arc oriented so the edge joins v to w, then

$$\epsilon(e) = \begin{cases} (-1)^{\mathrm{pos}(w:V)}, & v < w, \\ -(-1)^{\mathrm{pos}(v:V)}, & w < v. \end{cases} \tag{8}$$

- If n is even and e is a free edge or a mixed edge, then

$$\epsilon(e) = (-1)^{\mathrm{pos}(e:E)+|V_{\mathrm{free}}|+1}, \tag{9}$$

and if e is an arc, then

$$\epsilon(e) = (-1)^{\mathrm{pos}(\max\{v,w\})}. \tag{10}$$

An example of the differential is given in Fig. 3.

It is easy to see that δ does indeed raise the defect by 1 and leaves the order unchanged. This implies that the main degree of $\delta\Gamma$ is $|\delta\Gamma| = |\Gamma| + 1$. Thus, we can turn $\mathcal{LD}^*_*$ into a singly-graded complex (or differential graded algebra) where the term in grading g is given by

$$\bigoplus_{d,k:k(n-3)+d=g} \mathcal{LD}^d_k$$

(and similarly for $\mathcal{HD}^*_*$). Since each $\mathcal{LD}^d_k$ is finite-dimensional, each term as above is finite-dimensional.

3.2.2. *The shuffle product*

The shuffle product on the space of diagrams associated to knots was first considered in [8]. Here, we extend it to link diagrams as well as provide more details about its construction.

Consider two link diagrams $\Gamma_1 = (V(\Gamma_1), E(\Gamma_1), b_{\Gamma_1})$ and $\Gamma_2 = (V(\Gamma_2), E(\Gamma_2), b_{\Gamma_2})$. Let

$$\mathrm{seg}_i : V(\Gamma_i)_{\mathrm{seg}} \to \sqcup_i (I_i - \partial I_i)$$

be representatives of the equivalence class of the partition function for the segment vertices. Moreover, choose isomorphism class representatives for each diagram so

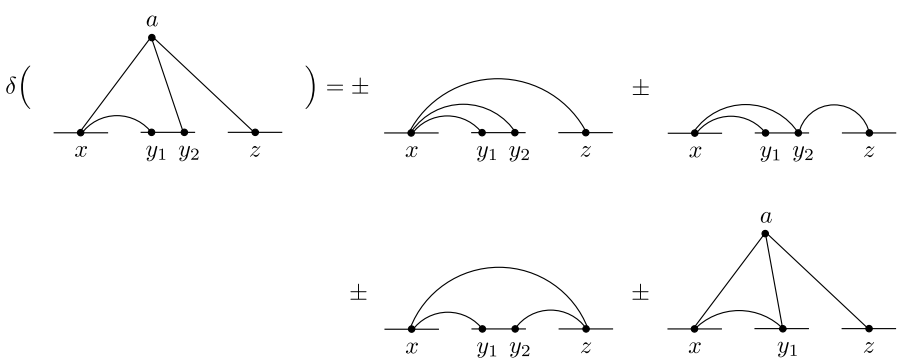

Fig. 3. An example of the differential. The signs depend on the parity of n.

that their vertex and edge sets are disjoint. Call an injective map

$$j : V(\Gamma_1)_{\text{seg}} \sqcup V(\Gamma_2)_{\text{seg}} \to \sqcup_i (I_i - \partial I_i)$$

admissible if its restriction to $V(\Gamma_i)_{\text{seg}}$ is in the same equivalence class as seg_i for $i = 1, 2$.

Definition 3.20. With Γ_1 and Γ_2 and an admissible map j as above, define

$$\Gamma_1 \cdot_j \Gamma_2 = (V(\Gamma_1 \cdot_j \Gamma_2), E(\Gamma_1 \cdot_j \Gamma_2), b_{\Gamma_1 \cdot_j \Gamma_2})$$

to be the diagram such that:

- The set $V(\Gamma_1 \cdot_j \Gamma_2) = V(\Gamma_1) \sqcup V(\Gamma_2)$;
- The set $E(\Gamma_1 \cdot_j \Gamma_2) = E(\Gamma_1) \sqcup E(\Gamma_2)$, and the orientations (if applicable) for edges are those induced by the orientations of elements of $E(\Gamma_1)$ and $E(\Gamma_2)$;
- The map $b_{\Gamma_1 \cdot_j \Gamma_2} = b_{\Gamma_1} \sqcup b_{\Gamma_2}$;
- The set $V(\Gamma_1 \cdot_j \Gamma_2)$ is decomposed as

  $$V(\Gamma_1 \cdot_j \Gamma_2)_{\text{seg}} \sqcup V(\Gamma_1 \cdot_j \Gamma_2)_{\text{free}},$$

  where

  - $V(\Gamma_1 \cdot_j \Gamma_2)_{\text{seg}} = V(\Gamma_1)_{\text{seg}} \sqcup V(\Gamma_2)_{\text{seg}}$, with ordering induced by the injection j,
  - $V(\Gamma_1 \cdot_j \Gamma_2)_{\text{free}} = V(\Gamma_1)_{\text{free}} \sqcup V(\Gamma_2)_{\text{free}}$, with ordering induced by the ordered pair $(V(\Gamma_1)_{\text{free}}, V(\Gamma_2)_{\text{free}})$, and hence
  - $V(\Gamma_1 \cdot_j \Gamma_2)$ is ordered by the ordered pair of ordered sets $(V(\Gamma_1 \cdot_j \Gamma_2)_{\text{seg}}, V(\Gamma_1 \cdot_j \Gamma_2)_{\text{free}})$;
- The ordering of $E(\Gamma_1 \cdot_j \Gamma_2)$ is that induced by the ordered pair $(E(\Gamma_1), E(\Gamma_2))$ of ordered sets.

Definition 3.21. For link diagrams Γ_1 and Γ_2, define their *shuffle product* $\Gamma_1 \bullet \Gamma_2$ by

$$\Gamma_1 \bullet \Gamma_2 = \sum_{[\text{admissible } j]} \epsilon(\Gamma_1, \Gamma_2) \Gamma_1 \cdot_j \Gamma_2, \tag{11}$$

where the sum is over equivalence classes of admissible maps, and where

$$\epsilon(\Gamma_1, \Gamma_2) = \begin{cases} (-1)^{|E(\Gamma_1)||V(\Gamma_2)_{\text{seg}}|}, & n \text{ even;} \\ 1, & n \text{ odd.} \end{cases}$$

From the definition of the main degree $|\Gamma|$, it is easy to see that $|\Gamma_1 \bullet \Gamma_2| = |\Gamma_1| + |\Gamma_2|$. Moreover, from straightforward unravelings of the definitions, one can prove the following propositions.

Proposition 3.22. *The shuffle product is graded-commutative; that is,*

$$\Gamma_1 \bullet \Gamma_2 = (-1)^{|\Gamma_1||\Gamma_2|} \Gamma_2 \bullet \Gamma_1.$$

Proposition 3.23. *The differential δ is a derivation with respect to the shuffle product. That is,*

$$\delta(\Gamma_1 \bullet \Gamma_2) = \delta(\Gamma_1) \bullet \Gamma_2 + (-1)^{|\Gamma_1|}\Gamma_1 \bullet \delta(\Gamma_2).$$

Hence we have the following proposition.

Proposition 3.24. *The diagram complex $(\mathcal{LD}, \delta, \bullet)$ is a commutative differential graded algebra (CDGA) with unit and its cohomology $\mathrm{H}^*(\mathcal{LD})$ is thus a commutative graded algebra.*

Remark 3.25. The authors of [8] use a different grading to make the shuffle product graded-commutative. It is easy to check that $|\Gamma|$ (which is the degree of the form Γ produces) agrees with their grading mod 2.

Remark 3.26. There is also a coproduct on $\mathcal{LD}$, analogous to the one given in [8]. Since we will not use this structure (shuffle product, on the other hand, will be needed in future work), we will only remark that this should give $\mathcal{LD}$ the structure of a Hopf algebra, and the map appearing in Theorem 4.33 induces a map of Hopf algebras in (co)homology.

3.3. *A subcomplex for the space of homotopy string links*

A homotopy string link need not be an embedding. As such, integration over $\mathcal{H}^n_m$ will not be possible in as general a way as prescribed on the complex $\mathcal{LD}$ (see Sec. 4.2 for more details) due to possible self-intersections of the components of the link. In this section, we will identify a subcomplex $\mathcal{HD}$ of $\mathcal{LD}$ for which it will be possible to carry out the integration and construct elements of $\Omega^*(\mathcal{H}^n_m)$.

Definition 3.27. Define the space of *homotopy link diagrams*, denoted $\mathcal{HD}$, to be the subspace of $\mathcal{LD}$ generated by diagrams Γ which:

(i) contain no loops; and
(ii) satisfy the condition that if there exists a path between distinct vertices on a given segment, then it must pass through a vertex on another segment.

Some examples of homotopy link diagrams are given in Fig. 4.

Proposition 3.28. *$\mathcal{HD}$ is a differential subalgebra of $\mathcal{LD}$.*

Proof. To show $\mathcal{HD}$ is a subcomplex of $\mathcal{LD}$, we must show that $\delta(\mathcal{HD}) \subset \mathcal{HD}$. Write $\Gamma = (V, E, b)$ and $\Gamma/e = (V', E', b')$, where $e = \{v, w\}$ with $v < w$. Suppose $\Gamma \in \mathcal{HD}$. We want to show each term Γ/e appearing in $\delta(\Gamma)$ is in $\mathcal{HD}$. Suppose to the contrary that Γ/e is not an element of $\mathcal{HD}$. There are two cases. The first case is that Γ/e has a loop. Then either (a) Γ itself has a loop or (b) Γ has a chord joining adjacent vertices on a segment or (c) Γ has multiple edges between a pair of vertices. Situations (a) and (b) are impossible since $\Gamma \in HLD$. In situation (c),

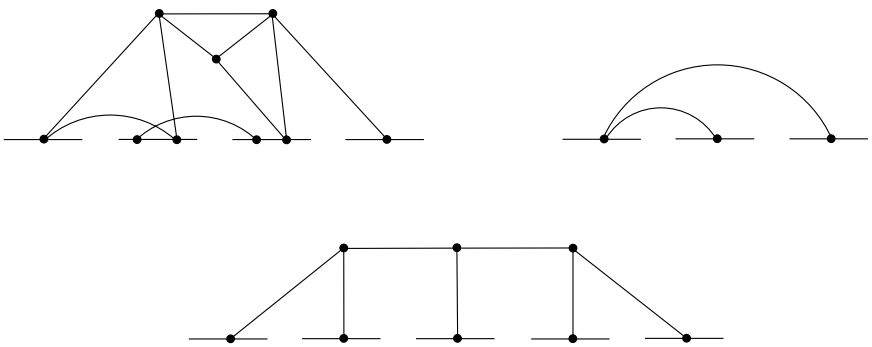

Fig. 4. Some examples of homotopy link diagrams (without decorations). The bottom one is a tree of the sort that will give rise to finite type invariants in Sec. 5.

Γ is set to zero, so $\delta(\Gamma)$ is also set to zero (so no terms Γ/e appear in $\delta(\Gamma)$). This covers the first case.

The second case is that v_1, v_2 are distinct segment vertices lying on the same segment of Γ/e, and there is a path $\alpha = \{e_i\}_{i=1}^k$ of edges from v_1 to v_2 which does not pass through a vertex on a different segment than the one on which v_1 and v_2 lie. In this case, it is enough to show that there is a path between vertices on the same segment in Γ which also does not pass through a vertex lying on a different segment.

Let $\alpha = \{e_i\}_{i=1}^k$ be a path in Γ/e as above, of minimal length. We have $v_1 \in b'(e_1), v_2 \in b'(e_k)$ and $b'(e_i) \cap b'(e_{i+1}) \neq \emptyset$ for all i. We may assume that v_1, v_2 are segment vertices in Γ, for otherwise e joins v_1 or v_2 to a segment vertex, and then $\{e, e_1, \ldots, e_k\}$ or $\{e_1, \ldots, e_k, e\}$ is a path in Γ joining vertices on the same segment without passing through another segment. Now if α has the property that $b(e_i) \cap b(e_{i+1}) \neq \emptyset$ for all i, then α itself is a path between v_1 and v_2 in Γ, contradicting the fact that $\Gamma \in \mathcal{HD}$. So let j be the smallest integer such that $b(e_j) \cap b(e_{j+1}) = \emptyset$. We have $b'(e_j) \cap b'(e_{j+1}) \neq \emptyset$, and $b' = R_{w \to v} \circ b$ for some v, w, so necessarily $w \in b(e_j)$ or $w \in b(e_{j+1})$. Without loss of generality assume $w \in b(e_j)$. Then it must be that $v \in b(e_{j+1})$, and in this case the edge e satisfies $b(e_j) \cap b(e) \neq \emptyset$ and $b(e_{j+1}) \cap b(e) \neq \emptyset$. Since α has minimal length, $\{e_1, \ldots, e_j, e, e_{j+1}, \ldots, e_k\}$ forms a path α' in Γ between v_1 and v_2 in Γ. We will be done if we can argue that w cannot be a segment vertex lying on a segment different from v_1 and v_2. But this is clear: if w is such a vertex, then v is such a vertex in Γ/e, and the original path α passes through this segment, a contradiction.

That $\mathcal{HD}$ is closed under the shuffle product is clear since this product does not create new paths of edges. □

A few words of clarification and justification for Definition 3.27 are in order. Our definition of $\mathcal{HD}$ excludes diagrams which contain a chord connecting two vertices on a single segment. It also excludes all possible diagrams which, via contractions

of edges, might produce such a chord. What we are trying to capture geometrically are linking phenomena which "ignore" the knotting of each strand. The reason for this is simple: there is no knotting of individual strands in $\mathcal{H}_m^n$, as they may pass through themselves. Once integration over diagrams is defined in Sec. 4.4, it will be clear that a chord between segment vertices captures something about linking between those segments. So when the segment vertices lie on the same segment, this means a chord between them captures something about self-linking, or knotting, of that segment. Similarly, integrals that correspond to loops will also only contain information about single strands.

3.4. *Diagram complexes in defect zero*

In Sec. 5, we will focus on the case $n = 3$ of classical links to see which link invariants (elements of $\mathrm{H}^0(\mathcal{L}_m^3)$ and $\mathrm{H}^0(\mathcal{H}_m^3)$) can be obtained via configuration space integrals from our diagram complexes. As we will see in Sec. 4.4, when $n = 3$, defect zero diagrams will correspond to degree zero forms (although in general the degree of forms corresponds to the main degree, not the defect), so we want

$$0 = 2|E(\Gamma)| - 3|V(\Gamma)_{\text{free}}| - |V(\Gamma)_{\text{seg}}|. \tag{12}$$

It was already noted in the discussion following Eq. (5) in Sec. 3.1 that these are precisely the trivalent diagrams.

Let

$$\mathrm{H}^0(\mathcal{LD}_k^*) := Z^0(\mathcal{LD}_k^*) := \ker(\delta : \mathcal{LD}_k^0 \to \mathcal{LD}_k^1)$$

denote the subspace of degree zero cocycles (i.e. degree zero cohomology classes) in the complex $\mathcal{LD}_k^*$. Similarly, let $\mathrm{H}^0(\mathcal{HD}_k^*)$ denote the subspace of degree zero cocycles (i.e. degree zero cohomology classes) in the complex $\mathcal{HD}_k^*$. (Note that in either case, for $n > 3$ this is not the same as the degree zero cocycles of the singly-graded complex graded by $|\Gamma|$.)

To understand the kernel of the differential $\delta : \mathcal{LD}_k^0 \to \mathcal{LD}_k^1$, we will examine the cokernel of its adjoint (i.e. dual) $\delta^* : (\mathcal{LD}_k^1)^* \to (\mathcal{LD}_k^0)^*$. Let

$$\mathrm{H}_0(\mathcal{LD}_k^*) := \operatorname{coker}(\delta^* : (\mathcal{LD}_k^1)^* \to (\mathcal{LD}_k^0)^*)$$

and similarly, let $\mathrm{H}_0(\mathcal{HD}_k^*)$ denote the corresponding cokernel in the complex $\mathcal{HD}_k^*$.

Let $|\mathrm{Aut}(\Gamma)|$ denote the size of the group of automorphisms of Γ as an unlabeled diagram. As defined at the end of Definition 3.8, these are automorphisms of graphs without any labels or edge orientations, but they must fix the segment vertices pointwise. Consider the inner product $\mathcal{LD}_k^d \otimes \mathcal{LD}_k^d \to \mathbb{R}$ which on diagrams is given by

$$\langle \Gamma_1, \Gamma_2 \rangle = \delta_{\Gamma_1,\Gamma_2} |\mathrm{Aut}(\Gamma_1)| (= \delta_{\Gamma_1,\Gamma_2} |\mathrm{Aut}(\Gamma_2)|),$$

where δ here is the Kronecker δ. This gives an isomorphism $\mathcal{LD}_k^d \xrightarrow{\cong} (\mathcal{LD}_k^d)^*$ via $\Gamma \mapsto \langle \Gamma, - \rangle$.

Thus, we can represent elements of $(\mathcal{LD}_k^d)^*$ by linear combinations of diagrams. We will write Γ^* for the element of $(\mathcal{LD}_k^d)^*$ which is the image of the diagram Γ under this isomorphism. For the rest of Sec. 3, a drawing of a diagram Γ will often mean the element Γ^*.

To understand $\delta^* : (\mathcal{LD}_k^1)^* \to (\mathcal{LD}_k^0)^*$, note that any diagram in $\mathcal{LD}_k^1$ has precisely one 4-valent vertex, as shown in the top of Figs. 5–7. In Figs. 5 and 7, i is a segment vertex, and in Fig. 6 it is a free vertex. It is not hard to see that the adjoint δ^* "blows up" 4-valent vertices in all possible ways, as shown in these three figures. In the first two figures, there are $\binom{4}{2}/2 = 3$ possibilities, corresponding to

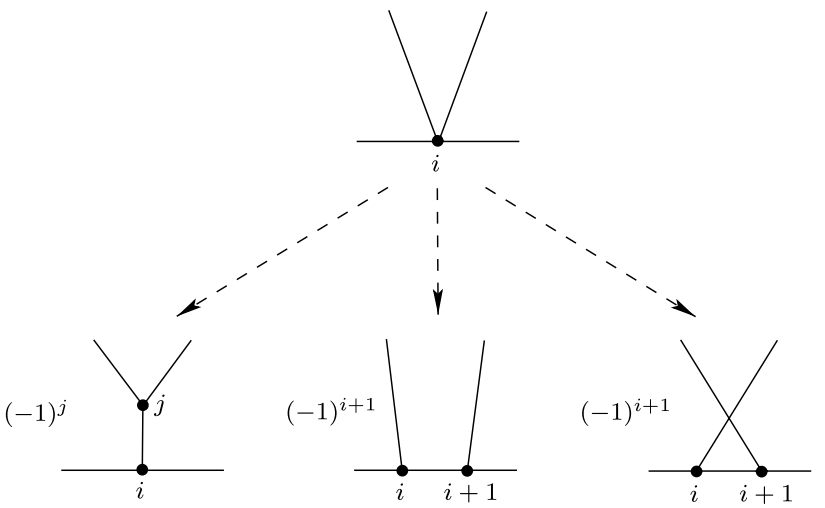

Fig. 5. Blowups giving rise to the *STU* relation.

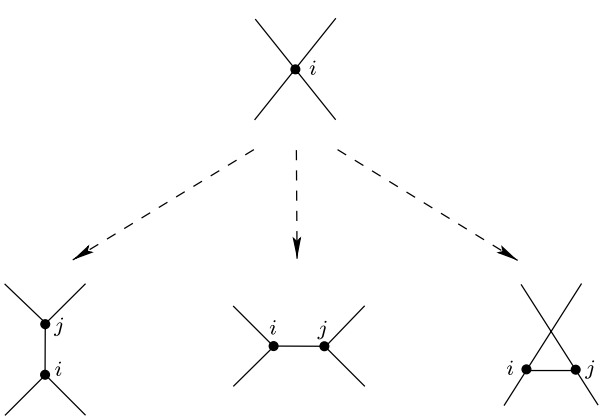

Fig. 6. Blowups giving rise to the *IHX* relation.

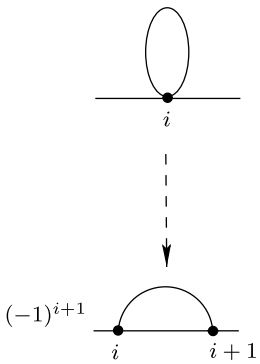

Fig. 7. Blowup giving rise to the $1T$ relation.

the possible ways of pairing four vertices. The image of δ^* is generated by three types of (linear combinations of) diagrams. Each type of generator is a sum of the diagrams shown in one of the figures with certain coefficients to be determined. The signs arise from the labeling conventions associated to edge contractions (in particular, recall that free vertices always have higher labels than segment ones, so $i < j$ in the left picture on the bottom of Fig. 5). In the first two of these figures, each diagram resulting from the blowup of a vertex is the same outside of the pictured portions as the other two diagrams in the triple.

Now we determine the coefficients in each sum that gives a generator of the image of δ^*. Call the three diagrams in Fig. 5 (with the indicated signs) S, T, and U. Call the three diagrams in Fig. 6 (with all signs $+1$) I, H, and X. Call the diagram in Fig. 7 $1T$.

Proposition 3.29. *The image of δ^* is generated by elements of the form*:

- $S^* + T^* + U^*$,
- $I^* + H^* + X^*$, *and*
- $(1T)^*$.

The statement of this proposition is certainly not new. For example, it appears in [17, Sec. 3]. It is a consequence of the following folklore result: Given a graph complex whose differential δ is a signed sum of edge contractions, the image of δ^* in the dual complex is the signed sum of edge expansions (i.e. with all coefficients ± 1), provided the duality is given by $\langle \Gamma, \Gamma' \rangle = \delta_{\Gamma,\Gamma'} |\mathrm{Aut}(\Gamma)|$. (For the $1T$ term, note that, by Definition 3.8, the diagram with the loop has the same number of automorphisms as the diagram with the isolated chord, i.e. for the parts pictured, there are no nontrivial automorphisms.) We could not find a proof of either this more general statement or the statement of Proposition 3.29 (the "unitrivalent case"), so we prove the proposition here.

Proof of Proposition 3.29. The proof amounts to checking that the linear combination we get from blowing up the 4-valent vertices in Figs. 5 (the STU case), 6 (the IHX case), and 7 (the $1T$ case) is in fact the one with all coefficients equal to 1.

The $1T$ *case*: Let L denote the 4-valent diagram in the top of Fig. 7. Here, the diagram $1T$ with the isolated chord is the only diagram whose image under δ contains L. (The only other potential such diagram is not in the complex, since by Definition 3.5, loops can only occur at segment vertices.) Thus,

$$\langle \delta^* L^*, \Gamma \rangle = \langle L, \delta\Gamma \rangle = \begin{cases} |\mathrm{Aut}(L)| & \text{if } \Gamma = 1T, \\ 0 & \text{for all other } \Gamma, \end{cases} \tag{13}$$

recalling that we take the pairing given by $\langle \Gamma_i, \Gamma_j \rangle = \delta_{ij}|\mathrm{Aut}(\Gamma_i)|$. Since $\mathrm{Aut}(L) \cong \mathrm{Aut}(1T)$, this implies that $\delta^* L^* = (1T)^*$.

The STU case: Let V denote the 4-valent diagram in the top of Fig. 5. Note that δ of any of the three diagrams S, T, U contains precisely one diagram V, and that these are the only diagrams whose image under δ can have such a term. Thus,

$$\langle \delta^* V^*, \Gamma \rangle = \langle V, \delta\Gamma \rangle = \begin{cases} |\mathrm{Aut}(V)| & \text{for all } \Gamma \in \{S, T, U\}, \\ 0 & \text{for all other } \Gamma, \end{cases} \tag{14}$$

There are two cases:

(i) All three diagrams S, T, U are in distinct isomorphism classes;
(ii) T is isomorphic to U.

Now note that in either case, $\mathrm{Aut}(T) \cong \mathrm{Aut}(U)$, since any automorphism must fix the free vertices attached to i and $i+1$, and the diagrams agree outside of the pictured part. Similarly, in either case, $\mathrm{Aut}(S) \cong \mathrm{Aut}(V)$ since an automorphism of S must fix the edge from i to j, and since S and V agree outside of that edge. We next analyze $\mathrm{Aut}(S)$. Under automorphisms of S, the vertices attached to j must either have singleton orbits or be in the same two-point orbit. Note that the case of singleton orbits corresponds precisely to case (i) above, while a two-point orbit corresponds to case (ii). In case (i), $\mathrm{Aut}(S) \cong \mathrm{Aut}(T)(\cong \mathrm{Aut}(U))$, for the same reason that $\mathrm{Aut}(T) \cong \mathrm{Aut}(U)$. Hence $|\mathrm{Aut}(V)| = |\mathrm{Aut}(\Gamma)|$ for all $\Gamma \in \{S, T, U\}$. Thus, Eq. (14) implies that in this case $\delta^* V^* = S^* + T^* + U^*$.

In case (ii), the index $[\mathrm{Aut}(S) : \mathrm{Aut}(S)_k] = 2$, where k is one of the vertices attached to j and where $\mathrm{Aut}(S)_k$ denotes the subgroup of $\mathrm{Aut}(S)$ fixing k. But $\mathrm{Aut}(S)_k \cong \mathrm{Aut}(T)$. So in this case $|\mathrm{Aut}(V)| = |\mathrm{Aut}(S)| = 2|\mathrm{Aut}(T)| = 2|\mathrm{Aut}(U)|$. So in this case, we conclude $\delta^* V^* = S^* + 2T^* = S^* + 2U^* = S^* + T^* + U^*$.

The IHX case: Let F be the 4-valent diagram in the top of Fig. 6. Let e be the edge pictured in any of the three diagrams I, H, X (by abuse of notation). In this IHX case, it is possible that δ of any of the diagrams $\Gamma = I, H, X$ has more than one term isomorphic to F. In fact, the number of such terms in $\delta\Gamma$ is given by the

size of the orbit of e (considered as an unordered pair) under the automorphism group of Γ. This number is $[\mathrm{Aut}(\Gamma) : \mathrm{Aut}(\Gamma)_e]$, where $\mathrm{Aut}(\Gamma)_e$ is the subgroup of automorphisms taking e to itself (the subgroup of $\mathrm{Aut}(\Gamma)$ which fixes Δ setwise). Thus, we have

$$\langle \delta^* F^*, \Gamma \rangle = \langle F, \delta\Gamma \rangle = \begin{cases} [\mathrm{Aut}(\Gamma) : \mathrm{Aut}(\Gamma)_e]|\mathrm{Aut}(F)| & \text{for all } \Gamma \in \{I, H, X\}, \\ 0 & \text{for all other } \Gamma. \end{cases} \tag{15}$$

This equation tells us that if we write $\delta^* F^* = \sum_\Gamma c_\Gamma \Gamma^*$, then we have

$$c_\Gamma |\mathrm{Aut}(\Gamma)| = [\mathrm{Aut}(\Gamma) : \mathrm{Aut}(\Gamma)_e]|\mathrm{Aut}(F)|.$$

We now analyze the index above using another subgroup of $|\mathrm{Aut}(\Gamma)|$. Let Δ be the subgraph of $\Gamma = I, H$, or X (again abusing notation) consisting of e, the edge joining i and j, and the edges incident to the endpoints of e. Let $\mathrm{Aut}(\Gamma)_\Delta$ denote the subgroup of $\mathrm{Aut}(\Gamma)$ which fixes every vertex of Δ. Clearly $\mathrm{Aut}(\Gamma)_\Delta < \mathrm{Aut}(\Gamma)_e$ since $\mathrm{Aut}(\Gamma)_\Delta$ fixes e (even as an ordered pair). So we can consider the index $[\mathrm{Aut}(\Gamma)_e : \mathrm{Aut}(\Gamma)_\Delta]$. This index is the order of the group $\mathrm{Aut}(\Gamma)_e|_\Delta$ of automorphisms in $\mathrm{Aut}(\Gamma)_e$ restricted to automorphisms of Δ; in other words, it is the group of restrictions to Δ of automorphisms of Γ that fix Δ setwise. Abbreviate this group G_Γ. Since G_Γ is a subgroup of

$$\mathrm{Aut}(\Delta) \cong \Sigma_2 \wr \mathbb{Z}/2 = \mathbb{Z}/2 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2) \cong D_4$$

(the group of symmetries of a square), the index in question is either $1, 2, 4$, or 8.

We divide our argument into two cases:

(1) all of I, H, X are distinct isomorphism classes;
(2) at least two of I, H, X are isomorphic.

Claim. *For $\Gamma \in \{I, H, X\}$, the indices $[\mathrm{Aut}(\Gamma)_e : \mathrm{Aut}(\Gamma)_\Delta] = |G_\Gamma|$ are all equal in case* (i); *in case* (ii), *two of the diagrams are isomorphic, and this index is twice as large for the third diagram as for either of the two isomorphic ones.*

Proof of Claim. Let $\sigma, \tau \in D_4$ denote the two elements that only swap two vertices (which as vertices of I, H, or X must be adjacent to the same endpoint of e). Let $\rho \in D_4$ be a rotation[a] by $\pi/2$. Identify D_4 with $\mathrm{Aut}(\Delta)$ via an embedding of Δ as shown in I.

First notice that $H \cong X$ if and only if at least one of σ, τ, ρ (or ρ^{-1}) is in G_I. One can also check that for any other element $\alpha \in D_4$ (meaning for $\alpha \in \{\rho^2, \rho\sigma, \sigma\rho\}$), we always have

$$\alpha \in G_I \Leftrightarrow \alpha \in G_H \Leftrightarrow \alpha \in G_X. \tag{16}$$

Thus, in case (i), none of $\sigma, \tau, \rho, \rho^{-1}$ is in G_I, and whatever remaining elements are in G_I are also in G_H and G_X. Interchaging the roles of I, H, X, we get that $G_I \cong G_H \cong G_X$, which is what we wanted to show.

[a]Note that this does *not* correspond to a rotation of any of the pictures of Δ in Fig. 6.

To finish case (ii), suppose $H \cong X$. Note first that none of $\sigma, \tau, \rho, \rho^{-1}$ can be an element of G_H or G_X; thus in this case, $|G_I| \geq 2|G_H|$ $(= 2|G_X|)$. However, if both σ and τ are elements of G_I, then their product $\sigma\tau$ $(= \tau\sigma = \rho^2)$ is in both G_H and G_X. This together with (16) implies $|G_I| \leq 2|G_H|$ $(= 2|G_X|)$. So in this case $|G_I| = 2|G_H| = 2|G_X|$. Interchanging the roles of I, H, X finishes the proof of the Claim in case (ii).

The right-hand side of Eq. (15) can be rewritten:

$$\begin{aligned}[\mathrm{Aut}(\Gamma) : \mathrm{Aut}(\Gamma)_e]|\mathrm{Aut}(F)| &= |\mathrm{Aut}(\Gamma)||\mathrm{Aut}(F)|/|\mathrm{Aut}(\Gamma)_e| \\ &= |\mathrm{Aut}(\Gamma)||\mathrm{Aut}(F)||\mathrm{Aut}(\Gamma)_\Delta|/|G_\Gamma|.\end{aligned}$$

So the coefficient c_Γ of Γ^* in $\delta^* F^*$ is equal to $(|\mathrm{Aut}(F)||\mathrm{Aut}(\Gamma)_\Delta|)/|G_\Gamma|$. Note that the groups $\mathrm{Aut}(\Gamma)_\Delta$ are isomorphic for all $\Gamma \in \{I, H, X\}$ since the diagrams agree outside of the pictures. Thus, the quantity in parentheses is independent of Γ.

By the Claim, we see that in case (i), the coefficients c_Γ are the same for all $\Gamma \in \{I, H, X\}$. Since we are working over $\mathbb{R}$, we can divide F by this number to get an equivalent generator with all c_Γ equal to one. In case (ii), we may suppose again without loss of generality that $H \cong X$. In this case, we showed that $|G_I| = 2|G_H| = 2|G_X|$. Thus, $2c_I = c_H = c_X$. So an appropriate multiple of $\delta^* F^*$ is equal to $I^* + 2H^* = I^* + 2X^* = I^* + H^* + X^*$. □

Proposition 3.29 implies that $\mathrm{H}_0(\mathcal{LD}_k^*)$ is the quotient of $\mathcal{LD}_k^0$ by all diagrams of the three types listed in its statement. Equivalently, $\mathrm{H}^0(\mathcal{LD}_k^*) = \ker\delta$ is for each k generated by trivalent diagrams such that the pairing with any of these three types of diagrams gives zero. The three types of relations by which we quotient to get $\mathrm{H}_0(\mathcal{LD}_k^*)$ are called the *STU relation*, the *IHX relation*, and the $1T$ *relation*. We will sometimes also use this terminology to describe the conditions that diagrams in $\mathrm{H}^0(\mathcal{LD}_k^*)$ must satisfy (see also Remark 3.34).

Remark 3.30. Bar-Natan [3] has shown that the *IHX* relation follows from the *STU* relation.

We now consider the case of $\mathrm{H}^0(\mathcal{HD}_k^*)$, where there are some additional observations to be made. First, the $1T$ relation is now vacuous since $\mathcal{HD}$ contains no diagrams with chords connecting vertices on the same segment. Second, suppose that the two loose edges in the top diagram of Fig. 5 belong to a loop of edges with all vertices except i free. This is depicted in Fig. 8.

Then blowing up vertex i can only result in one diagram, namely the (leftmost) diagram S^* from the *STU* relation. The other two would correspond to diagrams with paths between two segment vertices on the same segment that only go through free vertices, and such diagrams are not elements of $\mathcal{HD}$. We thus reduce the *STU* relation in $\mathrm{H}^0(\mathcal{HD}_k^*)$ to the condition that the diagram in Fig. 9 pairs to zero with any diagram in $\mathrm{H}^0(\mathcal{HD}_k^*)$.

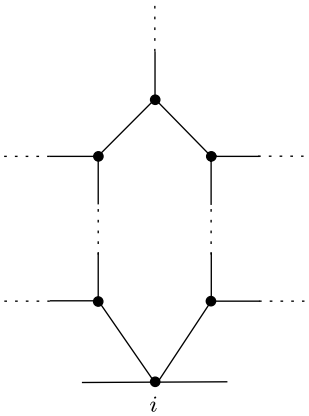

Fig. 8.

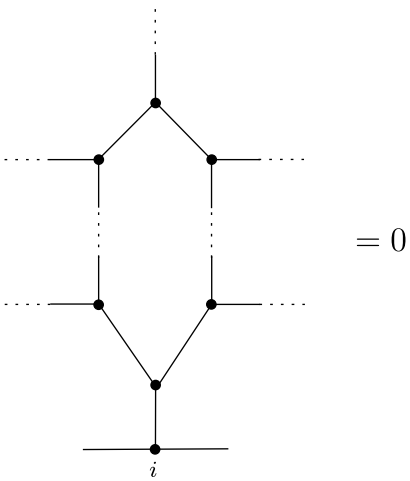

Fig. 9.

This relation extends to all diagrams with loops of free edges and not just those that are separated from a segment by a single mixed edge. Namely, the STU relation can be applied repeatedly to any path between the loop of free edges and a segment (there are always such paths since every free vertex must have a path to a segment vertex) and the situation can be reduced to that of Fig. 9. An example is given in Fig. 10, where "$= 0$" again means that this diagram pairs to zero with any diagram.

Remark 3.31. At first glance, it might seem that the diagram from Fig. 8 should not be permitted in $\mathcal{HD}$ since repeated contractions of its edges would eventually produce a loop at vertex i, and loops have been excluded from $\mathcal{HD}$. However, such

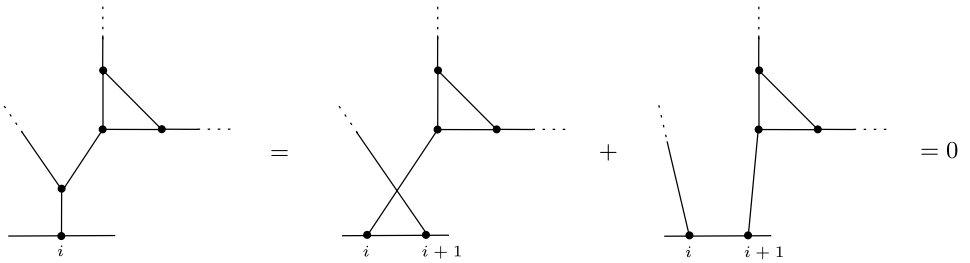

Fig. 10. An example of how a diagram with a loop of free edges pairs to 0 with any diagram.

contractions would first produce a double edge between vertex i and another free vertex, and a diagram with a double edge would already be zero by definition of $\mathcal{LD}$.

Remark 3.32. There is another interesting consequence of the STU relation in $\mathcal{HD}_k^0$ which we will have to use for in future work when we study Milnor Invariants in more detail. Namely, suppose that the same two loose edges in the top diagram of Fig. 5 end on the same segment. In other words, suppose the picture is as in Fig. 11, where the dots indicate that there might be other segment vertices between those pictured.

Then the STU relation obtained from blowing up this diagram is just $T^* + U^* = 0$, since the S diagram contains a path between two segment vertices that goes through only a free vertex. We thus get a special case of the STU relation in $\mathcal{HD}_k^0$, given in Fig. 12, i.e. the two sides of the equation are equal on all diagrams in $\mathcal{HD}_k^0$.

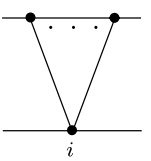

Fig. 11.

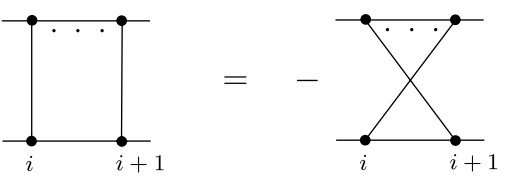

Fig. 12. A consequence of the STU relation in $\mathrm{H}^0(\mathcal{HD})$.

We now collect the observations made so far. Recall that we use the pairing on diagrams given by $\langle \Gamma_i, \Gamma_j \rangle = \delta_{ij} |\mathrm{Aut}(\Gamma_i)|$.

Proposition 3.33. *For each $k \geq 0$,*

- $\mathrm{H}^0(\mathcal{LD}_k^*)$ *consists of all linear combinations $\alpha = \sum a_i \Gamma_i$ of trivalent diagrams Γ_i which satisfy the*
  - *STU relation, i.e.* $(S^* + T^* + U^*)(\alpha) \equiv \langle S + T + U, \alpha \rangle = 0$,
  - *IHX relation, i.e.* $(I^* + H^* + X^*)(\alpha) \equiv \langle I + H + X, \alpha \rangle = 0$, *and*
  - $1T$ *relation, i.e.* $((1T)^*)(\alpha) \equiv \langle 1T, \alpha \rangle = 0$.
- $\mathrm{H}^0(\mathcal{HD}_k^*)$ *consists of all linear combinations $\alpha = \sum a_i \Gamma_i$ of trivalent diagrams Γ_i which satisfy the*
  - *STU relation and*
  - *IHX relation, and*
  - "$H1T$ *relation*", *which is that* $\Gamma^*(\alpha) = \langle \Gamma, \alpha \rangle = 0$ *for any* Γ *containing a closed path of edges.*

Remarks 3.34. (i) Regarding descriptions of $\mathrm{H}^0(\mathcal{LD}_k^*)$ and $\mathrm{H}^0(\mathcal{HD}_k^*)$ in previous literature, one difference is that Mellor [18] and Mellor–Thurston [20] work with the variant of $\mathcal{HD}_k^0$ consisting of unitrivalent diagrams without segments and without the STU relation (but they keep the other relations). In fact, the only reason we listed the last relation for $\mathrm{H}^0(\mathcal{HD}_k^*)$ (we could have left it out since it follows from the STU relation) is so that our description would exactly match those in [18, 20].

(ii) Define a *tree* to be a connected diagram such that there is a unique path of minimal length between any pair of distinct vertices, and define a *leaf* to be a mixed edge or chord of a tree (so a leaf has at least one associated segment vertex). Define a *forest* to be a diagram whose connected components are all trees. Since elements of $\mathcal{HD}_k^0$ are (sums of) trivalent diagrams without loops, every element is a sum of forests, each of whose trees has at most m leaves, where m is the number of distinct segments, and such that the segment vertices associated with the leaves all lie in distinct segments (that is, there is at most one segment vertex on each segment for a given tree in the forest). This was alluded to in the description of Fig. 4, where the bottom diagram is such a tree.

Definition 3.35. Define the space of *degree k link weight systems* $\mathcal{LW}_k$ as the vector space

$$((\mathcal{LD}_k^0)^*/(STU, IHX, 1T))^*,$$

where $(-)^*$ denotes the dual vector space, and where STU is the relation that $S^* + T^* + U^* = 0$, etc. Similarly, define the space of *degree k homotopy link weight systems* $\mathcal{HW}_k$ as the vector space

$$((\mathcal{HD}_k^0)^*/(STU, IHX, H1T))^*.$$

Since a vector space is canonically isomorphic to its double dual, we have the following.

Proposition 3.36. *There are canonical isomorphisms*

$$\mathcal{LW}_k \cong \mathrm{H}^0(\mathcal{LD}_k^*) \quad \textit{and} \quad \mathcal{HW}_k \cong \mathrm{H}^0(\mathcal{HD}_k^*).$$

Since $(\mathcal{LD}_k^0)^*$ and $(\mathcal{HD}_k^0)^*$ are spaces of diagrams, we can think of a weight system W as a functional on diagrams such that $W(S^* + T^* + U^*) = 0$, etc. This is how weight systems are typically defined, and this is how we will think of them in Sec. 5, where we will denote elements (diagrams) of $(\mathcal{LD}_k^0)^*$ and $(\mathcal{HD}_k^0)^*$ by letters without the superscript $*$.

Remarks 3.37. (i) The real reason we introduced the grading by order is that weight systems of order k are precisely finite type k invariants; see Theorems 5.6 and 5.8.

(ii) The above identification of weight systems with cocycles of diagrams can be used to reconcile integration from the graph complex with the integration of weight systems commonly found in the literature on finite type invariants. That is, the map $\mathrm{H}^0(\mathcal{LD}_k^*) \to \mathrm{H}^0(\mathcal{L}_m^3)$ can be thought of as a map $\mathcal{LW}_k \to \mathrm{H}^0(\mathcal{L}_m^3)$. We will discuss this in Sec. 5.

We make one last observation, which we will use in Sec. 5. We defined $\mathcal{LW}_k$ as the dual to a quotient of $(\mathcal{LD}_k^0)^*$ by certain relations. Instead of considering trivalent diagrams modulo these relations, one can reduce to the case of diagrams containing only chords, i.e. *chord diagrams*. That is, note that in $(\mathcal{LD}_k^0)^*/(STU, IHX, 1T)$, any trivalent diagram Γ can be rewritten as a sum of chord diagrams using the STU relation repeatedly. The resulting complex inherits a different relation as follows: Because the trivalent diagram in the STU relation can have both of its "loose" edges also ending in segments (necessarily different segments in the case of $\mathcal{HD}$), applying the STU relation twice gives what is known as the $4T$ relation, depicted in Fig. 13.

Denote by

$$\mathcal{LC}_k^0 \quad \text{and} \quad \mathcal{HC}_k^0$$

the $\mathbb{R}$-vector spaces generated by chord diagrams on m segments with k chords ending on $2k$ distinct vertices (since defect zero implies trivalence, two chords cannot end in a common segment vertex). For the latter space, there can be no chords with both endpoints on the same segment. We will call these the *link chord diagrams* and *homotopy link chord diagrams*. As in the case of trivalent diagrams, the duals $(\mathcal{LC}_k^0)^*$ and $(\mathcal{HC}_k^0)^*$ can be identified as spaces of chord diagrams. Using the relationship between the STU relation and the $4T$ relation, we have the following straightforward generalization of [3, Theorem 6].

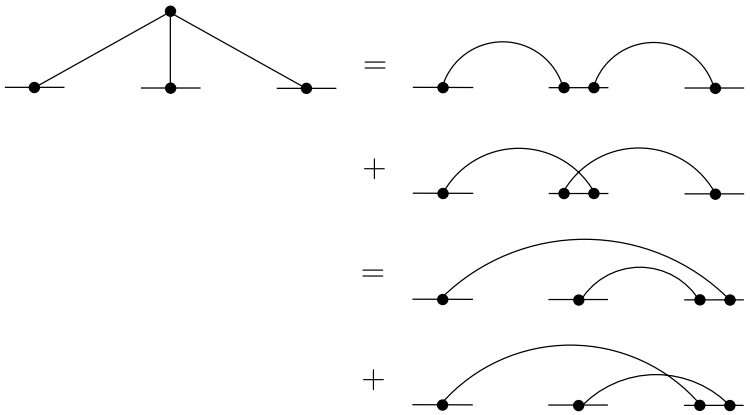

Fig. 13. Applying the STU relation to the middle and the right mixed edge produces the equality of the two pairs of chord diagrams. Any time four chord diagrams differ in two places as pictured, one obtains such an equality, called the $4T$ relation. The three arcs belong to distinct segments in the case of $\mathcal{HD}$ and some or all of them could belong to same segment in the case of $\mathcal{LD}$. An arbitrary permutation of the order of the three arcs in all the pictures is allowed.

Theorem 3.38. *There are isomorphisms*

$$(\mathcal{LD}_k^0)^*/(STU, IHX, 1T) \cong (\mathcal{LC}_k^0)^*/(4T, 1T) \quad \textit{and}$$

$$(\mathcal{HD}_k^0)^*/(STU, IHX, H1T) \cong (\mathcal{HC}_k^0)^*/4T.$$

Each isomorphism sends a diagram with no free vertices to itself and a diagram with free vertices to the sum of chord diagrams obtained from it via the STU relation.

Now denote by

$$\mathcal{LCW}_k \quad \text{and} \quad \mathcal{HCW}_k$$

the vector spaces of functionals on $(\mathcal{LC}_k^0)^*/(4T, 1T)$ and $(\mathcal{HC}_k^0)^*/4T$, respectively. Dualizing Theorem 3.38, we thus have isomorphisms

$$\mathcal{LW}_k \cong \mathcal{LCW}_k \quad \text{and} \quad \mathcal{HW}_k \cong \mathcal{HCW}_k. \tag{17}$$

Theorem 3.38 will be used in the proof of Theorem 5.8.

4. Configuration Space Integrals and Cohomology of Homotopy String Links

4.1. *Compactification of configuration spaces*

In this section, we review the standard construction of a compactification of configuration spaces over which we will integrate to produce invariants. This is necessary since integrals over the ordinary open configuration space may not converge. The original compactification is due to Fulton and MacPherson [9] and Axelrod and Singer [2].

Definition 4.1. For a manifold M, let

$$C(p, M) = \{(x_1, x_2, \ldots, x_p) \in M^p : x_i \neq x_j \text{ for } i \neq j\}$$

be the configuration space of p points in M. When $M = \mathbb{R}$, the configuration space has $p!$ components, and in this case $C(p, \mathbb{R})$ will mean the component consisting of those $(x_1, \ldots, x_p)$ such that $x_1 < \cdots < x_p$. Similarly, when $M = S^1$, $C(p, S^1)$ will mean one component where the points $x_1, \ldots, x_p$ are in a fixed cyclic order.

For a submanifold Y of a manifold X, the blowup $\mathrm{Bl}(X, Y)$ is the result of removing Y and replacing it by the sphere bundle of its normal bundle. Equivalently, this is the result of removing an open tubular neighborhood of Y.

Definition 4.2. For a compact manifold M, the (Fulton–MacPherson) compactification $C[p, M]$ is defined as the closure of the image of

$$C(p, M) \hookrightarrow M^p \times \prod_{S \subset \{1,\ldots,p\} | S| \geq 2} \mathrm{Bl}(M^S, \Delta_S),$$

where $\Delta_S = \{(x, x, \ldots, x) \in M^S\}$ is the thin diagonal in M^S. For $M = \mathbb{R}^n$, $C[p, \mathbb{R}^n]$ is considered as the subspace of $C[p+1, S^n]$ where the last point is fixed at ∞.

First, here are some general properties of $C[p, \mathbb{R}^n]$ that are relevant for our purposes. Proofs can be found in [28]:

(1) The space $C[p, \mathbb{R}^n]$ is a manifold with corners homotopy equivalent to $C(p, \mathbb{R}^n)$;
(2) The boundary of $C[p, \mathbb{R}^n]$ is given by points colliding or escaping to infinity;
(3) The directions and relative rates of collision are recorded, so that a k-stage collision (points coming together or going to infinity in k different stages rather than all of them doing this at the same instance) gives a point in a codimension k stratum of $C[p, \mathbb{R}^n]$. These k stages are the *screens* explained below.

The last property in particular says that codimension-one faces of $C[p, \mathbb{R}^n]$ consist of configurations where some subset of the points has come together or escaped to infinity at the same time. These faces are of particular interest since they play a role in checking whether some differential form obtained on the space of links is closed (i.e. they are relevant for an application of Stokes' Theorem).

Some elaboration is necessary in order to define configuration space integrals for string links. A stratum of $C[p, M]$ is labeled by a collection $\{S_1, \ldots, S_k\}$ of distinct subsets $S_i \subset \{1, \ldots, p\}$ with $|S_i| \geq 2$ and satisfying the condition

$$S_i \cap S_j \neq \emptyset \Rightarrow \text{ either } S_i \subset S_j \text{ or } S_j \subset S_i.$$

In other words, the S_i are pairwise nested or disjoint. For each set S_i in the collection, we can think of the points in S_i as having collided. If there is an $S_j \subset S_i$ in the collection, we can think of the points in S_j as having first collided with each other and then with the remaining points in S_i. Two strata indexed by $\{S_1, \ldots, S_k\}$

and $\{S'_1, \ldots, S'_j\}$ intersect precisely when the set $\{S_1, \ldots, S_k, S'_1, \ldots, S'_j\}$ satisfies the above condition. In that case, that is the set which indices the intersection.

Roughly speaking, each S_i corresponds to an "infinitesimal configuration" or *screen*, and all the screens together encode directions and relative rates of collision, as follows. Let $s_i = |S_i|$. In the case where all S_i are disjoint, the screen corresponding to S_i is a point $\vec{u}_{S_i} \in (C(s_i, T_xM))/(\mathbb{R}^n \rtimes \mathbb{R}_+)$, where $\mathbb{R}^n \rtimes \mathbb{R}_+$ is the group of translations and (oriented) scalings of $\mathbb{R}^n \cong T_xM$. In the case where all the S_i are nested, say as $S_1 \subset \cdots \subset S_k$, the screen $\vec{u}_{S_i}$ is a point in $(C(s_i - s_{i-1} + 1, T_xM))/(\mathbb{R}^n \rtimes \mathbb{R}_+)$ (where we set $s_0 = 0$). In general, $\vec{u}_{S_i}$ is a configuration of points in T_xM, modulo the action of $(\mathbb{R}^n \rtimes \mathbb{R}_+)$. Each of the p points in a limiting configuration (i.e. a configuration in the boundary of $C[p, \mathbb{R}^n]$) corresponds to a point in possibly multiple screens; if the point is indexed by $j \in \{1, \ldots, p\}$, it corresponds precisely to one point in each S_i that contains j. The number of points in $\vec{u}_{S_i}$ is obtained by taking the points in S_i and, for each maximal proper S_j contained in S_i, replacing the points in S_j by a single point; i.e. all the points in $\vec{U}_{S_i}$ become one point in $\vec{U}_{S_j}$ when $S_i \subset S_j$. From this description, one can verify that the stratum labeled $\{S_1, \ldots, S_k\}$ has codimension k. Again, more precise details can be found in [2, 9].

Remark 4.3. An alternative but equivalent definition of this compactification was given by Sinha in [28], which is as follows. Suppose M is a compact submanifold of $\mathbb{R}^N$. Then for all $1 \leq i < j < k \leq p$ we have maps

$$v_{ij} = \frac{x_j - x_i}{|x_j - x_i|} \in S^{N-1}, \quad a_{ijk} = \frac{|x_i - x_j|}{|x_i - x_k|} \in [0, \infty], \tag{18}$$

whose domain is $C(p, M)$ and where $[0, \infty]$ denotes the one-point compactification of $[0, \infty)$. These maps measure the direction and relative rates of collision of configuration points, respectively. Adding this information to the configuration space is achieved by considering the map

$$\begin{aligned} \gamma : C(p, M) &\to M^p \times (S^{N-1})^{\binom{p}{2}} \times [0, \infty]^{\binom{p}{3}}, \\ (x_1, \ldots, x_p) &\mapsto (x_1, \ldots, x_p, v_{12}, \ldots, v_{ij}, \ldots, v_{(p-1)p}, \\ &\qquad a_{123}, \ldots, a_{ijk}, \ldots, a_{(p-2)(p-1)p}). \end{aligned} \tag{19}$$

The closure of the image of γ turns out to be diffeomorphic (as a manifold with corners) to $C[p, M]$. That is,

$$C[p, M] = \overline{\gamma(C(p, M))} \subset M^p \times (S^{N-1})^{\binom{p}{2}} \times [0, \infty]^{\binom{p}{3}}.$$

Since $C[p, \mathbb{R}^n]$ is defined via $C[p+1, S^n]$, we would have to take N above to be $n+1$ if we were to use this definition. Then the unit vector difference maps v_{ij} would land in S^n, rather than the more geometrically obvious candidate, S^{n-1}. (If one tries to use S^{n-1}, the maps v_{ij} cannot extend from $\mathbb{R}^n$ to S^n; this is why Definition 4.2 is better suited for our purposes.)

4.2. *Bundles of compactified configuration spaces*

Given $\Gamma \in \mathcal{LD}$, we will construct in this section a certain bundle of configuration spaces over $\mathcal{L}^n_m$. There is already a standard recipe for doing this which was initiated in the case of closed knots ($m = 1$) in [5] and fully developed in [7]. Generalizing this recipe to long knots or closed links is straightforward. In generalizing to homotopy string links, two issues arise. First, some care needs to be taken to extend this construction to ordinary string (i.e. long) links. The second and perhaps more serious issue is that even after extending to string links, this construction fails to even produce a bundle over $\mathcal{H}^n_m$ by restriction to the subcomplex $\mathcal{HD}$ as we will see in Sec. 4.2.4.

Resolving the first issue essentially just relies on our definition of string links, in which different components approach infinity in different directions (see Definition 2.1), as well as properties of the Fulton–MacPherson compactification. We take the standard bundle construction, as in [7, 31], as our starting point, and we describe how to make the construction work for string links in Sec. 4.2.1.

To fix the second issue, we devise a more refined way of constructing bundles which works over both $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$. In Sec. 4.2.4, we refine this construction to produce bundles over spaces of homotopy string links. The difference between the two approaches can be summarized very succinctly: in the standard approach, only vertices of a diagram are taken into account in the construction of bundles, whereas in the new approach, we will take into account both vertices and edges. We will show the compatibility of the approaches in Sec. 4.4.

4.2.1. *Bundles of compactified configuration spaces from vertices of a diagram*

A diagram $\Gamma \in \mathcal{LD}$ will define a configuration space where the segment vertices of Γ correspond to points moving along a link in $\mathbb{R}^n$ and free vertices correspond to points that are free to move anywhere in $\mathbb{R}^n$.

Suppose $\Gamma \in \mathcal{LD}$ has i_j segment vertices on the jth segment, $1 \le j \le m$, and s free vertices. For any link $L \in \mathcal{L}^n_m$, the evaluation map

$$\mathrm{ev}_\Gamma(L)\colon \prod_{j=1}^{m} C(i_j, \mathbb{R}) \to C\left[\sum_{j=1}^{m} i_j, \mathbb{R}^n\right] \tag{20}$$

is given by evaluating the jth strand of L on i_j configuration points. In other words, it is given by

$$\begin{aligned}(L, (x^1_1, \ldots, x^1_{i_1}), \ldots, (x^m_1, \ldots, x^m_{i_m}))\\ \mapsto (L(x^1_1), \ldots, L(x^1_{i_1}), \ldots, L(x^m_1), \ldots, L(x^m_{i_m})).\end{aligned}$$

Let $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ denote the closure of the image of $\mathrm{ev}_\Gamma(L)$, where we think of $\vec{i}$ as $(i_1, \ldots, i_m)$. Suppressing the dependence on L will be justified by the next lemma.

So far it is clear that for any $L \in \mathcal{L}^n_m$, $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ is compact and that its interior is diffeomorphic to $\prod_{j=1}^m C(i_j, \mathbb{R})$.

Lemma 4.4. *For any $L \in \mathcal{L}^n_m$, the space $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ has the structure of a manifold with corners, independent of L.*

Proof. We will show that the manifold with corners structure comes from that on $C[i_1 + \cdots + i_m + 1, S^n]$. First, note that all the added limit points are in the boundary of $C[i_1 + \cdots + i_m, \mathbb{R}^n]$. Around such a point, a neighborhood in $C[i_1 + \cdots + i_m, \mathbb{R}^n]$ has various strata, points of which are described by collections of screens, as was explained after Definition 4.2. To describe the corresponding neighborhood in $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$, we replace these spaces of screens by similar spaces which have lower dimension, but will have the same codimension in the latter space.

At a collision of $s \geq 2$ points at a point x away from ∞, the space $C(s, T_x\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+)$ is replaced by $C(s, T_xL)/(\mathbb{R} \rtimes \mathbb{R}_+)$, whereby abuse of notation L also denotes the image of L and where $\mathbb{R} \rtimes \mathbb{R}_+ < \mathbb{R}^n \rtimes \mathbb{R}_+$ is the subgroup of translations and scalings of $T_x\mathbb{R}^n$ which take T_xL to itself.

Consider a collision of ≥ 1 points with ∞, first as just a configuration in $C[I, \mathbb{R}^n] \subset C[I + 1, S^n]$, where $I := i_1 + \cdots + i_m$. Consider a stratum incident to that configuration, labeled by $\{S_1, \ldots, S_k\}$. Let S_i be a set containing the $(I+1)$th point ∞, and let $s_i = |S_i|$. We can describe the "screen-space" corresponding to S_i as $C(s_i, T_\infty S^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+) \cong C(s_i - 1, T_\infty S^n \backslash \{0\})/\mathbb{R}_+$, where this identification comes from fixing the $(I + 1)$th point ∞ at the origin in $T_\infty S^n$.

Now suppose that the points in the configuration are on the link, so that this configuration is also in $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$. We can describe a neighborhood in that space by replacing the screen above by a configuration of points in $T_\infty S^n \backslash \{0\}$ which are constrained to lie in certain open rays emanating from the origin, modulo scaling. These rays correspond to the directions of the fixed linear embedding. In other words, we replace $C(s_i - 1, T_\infty S^n \backslash \{0\})/\mathbb{R}_+$ by $C(s_i - 1, T_\infty L \backslash \{0\})/\mathbb{R}_+$. (Of course, the one-point compactification of L is not a manifold at $\infty \in S^n$, but $T_\infty L$ seems like appropriate notation for the subset of lines through the origin in $T_\infty S^n$ corresponding to the components of L.) This treatment of collisions at infinity is where we use our string link components to have different directions toward infinity, as required in Definition 2.1.

Note that not all the strata in $C[I, \mathbb{R}^n]$ occur as strata in $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ because points in different components of the link cannot collide away from ∞ (and furthermore, if a point on a link component has its two neighbors approaching ∞, then it must approach ∞ too). But for a given $\mathcal{S} = \{S_1, \ldots, S_k\}$ which does index a stratum $\mathfrak{S}$ of $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$, any subset of $\mathcal{S}$ clearly indices a stratum in $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$. These subsets correspond precisely to the higher-dimensional strata which intersect a neighborhood of any point in $\mathfrak{S}$. This is the sense in which the corner structure on $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ is inherited from the one on $C[I, \mathbb{R}^n]$. Hence the space $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ can be seen to be a manifold with corners for the same reason that $C[I, \mathbb{R}^n]$ is.

In more detail, we parametrize a neighborhood of a codimension k point in $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$, just as is done for $C[i, \mathbb{R}]$ in [30, Sec. 4.1] or for $C[i, M]$ in [2, Sec. 5.4]:

Let $\mathfrak{S}$ be a stratum of codimension k, indexed by a collection $\mathcal{S}$ of subsets $S_1, \ldots, S_k$ of $\{1, 2, \ldots, 1 + \sum_j i_j\}$ (with the last point here corresponding to ∞). A point c in such a stratum is described by (not necessarily distinct) points $x_1 = x_1(c), \ldots, x_p = x_p(c) \in S^n = \mathbb{R}^n \cup \{\infty\}$, together with k screens $\vec{u}_S$ (one for each $S \in \mathcal{S}$) at some of these x's, with possibly multiple screens at any given x. A screen $\vec{u}_S$ away from ∞ consists of $u_{S,1} < u_{S,2} < \cdots \in \mathbb{R} \cong T_x L$ such that $\sum_h u_{S,h} = 0$ and $\sum_h |u_{S,h}|^2 = 1$. A screen $\vec{u}_S$ at ∞ describes the escape to ∞ of $a_j + b_j$ points on the jth strand, a_j of them in the "negative direction" and b_j of them in the "positive direction". Such a screen is given by

$$(u^1_{S,1} < \cdots < u^1_{S,a_1}, \ldots, u^m_{S,1} < \cdots < u^m_{S,a_m};$$
$$v^1_{S,1} < \cdots < v^1_{S,b_j}, \ldots, v^m_{S,1} < \cdots < v^m_{S,b_m}),$$

where $u^j_{S,h} \in (-\infty, 0)$ and $v^j_{S,h} \in (0, \infty)$ are points in the two rays in $T_\infty L \backslash \{0\}$ coming from the jth component of the link, and where these parameters satisfy

$$\sum_j \left(\sum_{h=1}^{a_j} |u^j_{S,h}|^2 + \sum_{h=1}^{b_j} |v^j_{S,h}|^2 \right) = 1.$$

Note that either type of screen $\vec{u}_S$ is given by as many parameters as there are elements in S. Using the set of the x's in $\mathbb{R}^n = S^n \backslash \{\infty\}$ (without multiplicity) and the parameters in the $\vec{U}_S$, we can parametrize an open neighborhood $V \subset \mathfrak{S}$ of an interior point $c_0 \in \mathrm{int}(\mathfrak{S})$, showing that $\mathfrak{S}$ is a manifold (of dimension $np - k$).

Thus, to understand the corner structure of $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$, it suffices to provide a map from an open neighborhood $U \times [0, \epsilon)^k$ of $(c_0, 0)$ in $\mathrm{int}(\mathfrak{S}) \times [0, \infty)^k$ to $C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$. We first define a map $U \times (0, \epsilon)^k \to (S^n)^{i_1 + \cdots + i_m}$ by

$$(c, r_1, \ldots, r_k) \mapsto \left(\exp_{x_1(c)} \left(\sum_{\ell \in \{1, \ldots, k\} : S_\ell \ni 1} \widetilde{r}_\ell u_{S_\ell, 1} \right), \ldots, \right.$$
$$\left. \exp_{x_p(c)} \left(\sum_{\ell \in \{1, \ldots, k\} : S_\ell \ni p} \widetilde{r}_\ell u_{S_\ell, p} \right) \right),$$

where $\exp_x$ is the exponential map $T_x L \to S^n$ and where

$$\widetilde{r}_\ell = \prod_{\ell' : S_{\ell'} \supset S_\ell} r_{\ell'}.$$

Even though $T_\infty L$ is strictly not a tangent space to a manifold, it has an exponential map coming from the restriction of the exponential map from $T_\infty S^n$. For a sufficiently small neighborhood $U = U(c_0)$ and sufficiently small $\epsilon = \epsilon(c_0)$, one can show that this map is injective. The map above is essentially [2, Eq. 5.71], and the proof of injectivity is essentially the same as the proof given in that reference.

Finally, this map extends continuously to a map $U \times [0,\epsilon)^k \to C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ by mapping a point $(c,0)$ into $\mathfrak{S}$ and, more generally, by mapping a boundary point $(c,\vec{r})$ into the stratum indexed by $\{S_\ell : r_\ell = 0\}$. □

It is now clear that for any L this gives a compactification of $\prod_{j=1}^m C(i_j,\mathbb{R})$ whose manifold-with-corners structure is independent of L. So we can write

$$\mathrm{ev}_\Gamma : \mathcal{L}_m^n \times C\left[\vec{i}; \coprod_{j=1}^m \mathbb{R}\right] \to C\left[\sum_{j=1}^m i_j, \mathbb{R}^n\right]. \tag{21}$$

Returning to the ordinary compactified configuration spaces, we have the projection

$$\mathrm{pr} : C\left[\sum_{j=1}^m i_j + s, \mathbb{R}^n\right] \to C\left[\sum_{j=1}^m i_j, \mathbb{R}^n\right] \tag{22}$$

given by forgetting the last s points of a configuration, as well as all the v_{ij} and a_{ijk} which involve any of the last s points.

Definition 4.5. Given $\Gamma \in \mathcal{LD}$ with i_j segment vertices on the jth segment and s free vertices, let $\vec{\imath} = (i_1, \ldots, i_m)$, and let

$$C[\vec{\imath} + s;\, \mathcal{L}_m^n, \Gamma]$$

be the pullback of pr along ev_Γ:

$$\begin{array}{ccc} C[\vec{\imath} + s; \mathcal{L}_m^n, \Gamma] & \longrightarrow & C[\sum_{j=1}^m i_j + s, \mathbb{R}^n] \\ \downarrow & & \downarrow \mathrm{pr} \\ \mathcal{L}_m^n \times C[\vec{i}; \coprod_{j=1}^m \mathbb{R}] & \xrightarrow{\mathrm{ev}_\Gamma} & C[\sum_{j=1}^m i_j, \mathbb{R}^n] \end{array} \tag{23}$$

We then have the following special case of [5, Proposition A.3].

Proposition 4.6. *With Γ as above, the projection*

$$\overline{\pi}_{\mathcal{L},\Gamma} : C[\vec{\imath} + s;\, \mathcal{L}_m^n, \Gamma] \to \mathcal{L}_m^n$$

is a smooth fiber bundle whose fiber is a finite-dimensional smooth manifold with corners.

Returning to the perspective of diffeology (as explained in Sec. 2.2), it is convenient to think of this as a compatible collection of bundles, one for each $\psi : M \to \mathcal{L}_m^n$, just as the one above but with $\mathcal{L}_m^n$ replaced by M. Here, ψ is a smooth map and M is a finite-dimensional manifold (without corners). In that case, it is not hard to generalize the proof of [5, Proposition A.3] from one fiber to the whole bundle. It is also not hard to see that the projection map $\overline{\pi}$ is smooth. (Note that the corner

structure plays no role in the smoothness of $\overline{\pi}$, since $d\overline{\pi}$ sends all the tangent vectors orthogonal to boundary faces to zero.)

We will denote the fiber of $\overline{\pi}_{\mathcal{L},\Gamma}$ over a link L by

$$\overline{\pi}_{\mathcal{L},\Gamma}^{-1}(L) = C[\vec{i} + s; L, \Gamma].$$

We think of this space as a configuration space whose first i_1 points must lie on the first strand of L, second i_2 must lie on the second strand, and so on, while the last s is free to move anywhere in $\mathbb{R}^n$ (including on the image of L).

4.2.2. *Bundles from diagram vertices and a difficulty with homotopy links*

If Γ is a diagram in $\mathcal{HD}$, then the above construction will not in general produce a fiber bundle over $\mathcal{H}_m^n$. The first problem is that a generic element $H \in \mathcal{H}_m^n$ need not be an embedding or even an immersion, so that the target of the evaluation map is not the usual compactified configuration space, but rather a "partial" configuration space where some points are allowed to collide (without regard for how), while others are not. The second problem, not as easily overcome, is that the map from one partial configuration space to another which restricts to some subset of the original set of points is usually not a fibration, making it difficult to produce a fiber bundle by pullback. As an illustration, consider the following example.

Example 4.7. Define

$$C(2, 1; \mathbb{R}^n) = \{(x_1, x_2, y) \in (\mathbb{R}^n)^3 : x_1, x_2 \neq y\}$$

and let $C[2, 1; \mathbb{R}^n]$ denote its compactification (we only compactify along the diagonals which have been removed). Next, take $m = 1$ (so there is one strand) and any value of n, and consider the evaluation map

$$\mathrm{ev} : \mathcal{H}_1^n \times C[2, \mathbb{R}] \to \mathbb{R}^n \times \mathbb{R}^n.$$

The projection

$$\mathrm{pr} : C[2, 1; \mathbb{R}^n] \to \mathbb{R}^n \times \mathbb{R}^n$$

to the first two coordinates is not even a fibration, as the fiber over a point (x_1, x_2) with $x_1 = x_2$ is homotopy equivalent to S^{n-1}, while the fiber over such a pair with $x_1 \neq x_2$ is homotopy equivalent to $S^{n-1} \vee S^{n-1}$. The problem persists with links of more components.

However, if we only allow one point on each strand for the evaluation map, then we can proceed as follows. We have an evaluation map (where $\vec{1} := (1, 1, \ldots, 1)$)

$$\mathrm{ev} : \mathcal{H}_m^n \times C\left[\vec{1}; \coprod_{j=1}^{m} \mathbb{R}\right] \to C[m, \mathbb{R}^n]$$

obtained by evaluating each strand of a homotopy link on exactly one point in that strand. The image necessarily lies in the interior of the compactified configuration space $C[m, \mathbb{R}^n]$ since the images of the m strands are disjoint.

We again have a projection map

$$\mathrm{pr}: C[m+s, \mathbb{R}^n] \to C[m, \mathbb{R}^n] \tag{24}$$

which is a fibration (of manifolds with corners) so that one can form the pullback

$$\begin{tikzcd} C[\vec{1}+s; \mathcal{H}_m^n] \arrow[r] \arrow[d] & C[m+s, \mathbb{R}^n] \arrow[d, "\mathrm{pr}"] \\ \mathcal{H}_m^n \times C[\vec{1}; \coprod_{j=1}^m \mathbb{R}] \arrow[r, "\mathrm{ev}"] & C[m, \mathbb{R}^n] \end{tikzcd} .$$

There is now a bundle

$$C[\vec{1}+s; \mathcal{H}_m^n] \to \mathcal{H}_m^n \tag{25}$$

for the same reason we have one in Proposition 4.6. (It should be noted that A.3 of [5] may appear to the reader not to apply, but it depends on A.5, which does apply in this situation and gives the result we claim.) We now use this observation to build bundles over $\mathcal{H}_m^n$ for any diagram $\Gamma \in \mathcal{HD}$, and this will naturally extend to diagrams in $\mathcal{LD}$. In order to do so, we need to break our diagrams up into pieces, called "grafts".

4.2.3. *The graft components of a diagram*

Definition 4.8. For a vertex v in a diagram Γ, let $N(v)$ be the set of all pairs (w, e) such that $b(e) = \{v, w\}$.

Thus, $N(v)$ consists of all the "neighbors" of v counted with multiplicity according to edges.

Definition 4.9. Let $\Gamma = (V, E, b) \in \mathcal{LD}$ be a diagram. Define the *hybrid* of Γ to be the diagram $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{E}, \widetilde{b})$ defined as follows: The set $\widetilde{V}$ is obtained from V by replacing each segment vertex $v \in V$ of Γ with the set $v \times N(v)$, the elements of which will represent new vertices, and otherwise the vertex set is unchanged. The edge set $\widetilde{E}$ is equal to E. The map $\widetilde{b}$ is induced from b according to the following rule: Suppose $b(e) = \{v, w\}$. If $v, w \in \widetilde{V}$, then $\widetilde{b}(e) = b(e)$. If one of v or w, say v, is a segment vertex, then $\widetilde{b}(e) = \{(v, (w, e)), w\}$. If both are, then $\widetilde{b}(e) = \{(v, (w, e)), (w, (v, e))\}$.

The hybrid is not a link diagram, but it does induce certain link diagrams which are subdiagrams of the original link diagram Γ.

Definition 4.10. For a diagram $\Gamma \in \mathcal{LD}$ with hybrid $\widetilde{\Gamma}$, define the *graft components* of $\widetilde{\Gamma}$ to be the set of path components (i.e. connected components) of $\widetilde{\Gamma}$.

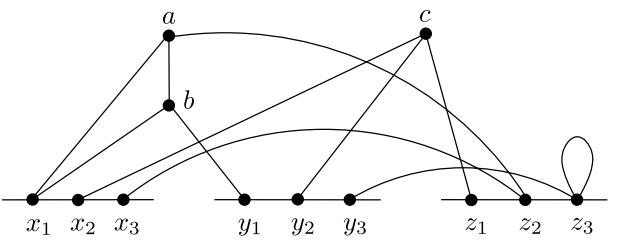

Fig. 14.

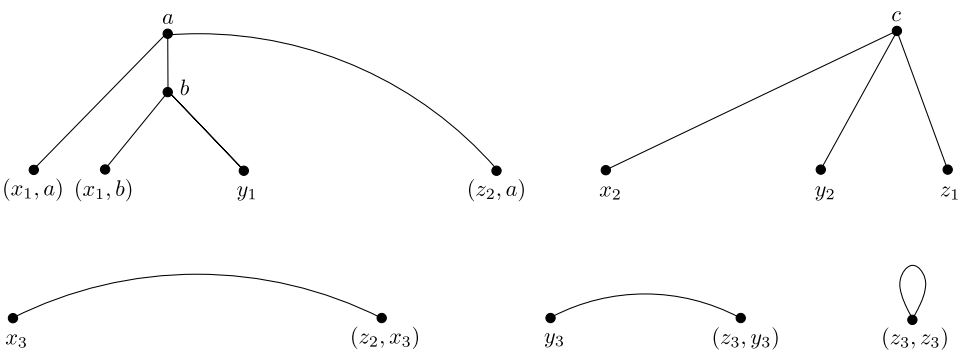

Fig. 15. The five graft components of the diagram in Fig. 14. We have simplified the labels on the vertices of the graft components because the original diagram does not possess multiple edges between a given pair of vertices.

Example 4.11. Consider the diagram Γ in Fig. 14. The five graft components of its hybrid $\widetilde{\Gamma}$ are given in Fig. 15.

The following is clear by construction.

Proposition 4.12. *Each chord of Γ gives rise to a graft component consisting of two vertices and a single edge, and each loop at a segment vertex gives rise to a graft component with a single vertex and a single edge.*

Although the hybrid $\widetilde{\Gamma}$ is not a link diagram, each graft component $c(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$ canonically defines an element of $\mathcal{LD}$, with its structure induced by Γ.

Definition 4.13. Suppose the diagram $c(\widetilde{\Gamma}) = (V(c(\widetilde{\Gamma})), E(c(\widetilde{\Gamma})), b_{c(\widetilde{\Gamma})})$ is a graft component of $\widetilde{\Gamma}$, so that $V(c(\widetilde{\Gamma})) \subset \widetilde{V}$ and $E(c(\widetilde{\Gamma})) \subset \widetilde{E} = E$. The forgetful map $\widetilde{V} \to V$ identifies $c(\widetilde{\Gamma})$ with a subdiagram $c(\Gamma)$ of Γ, called a *graft* of Γ which inherits all the necessary structure for it to define an element of $\mathcal{LD}$.

If $\Gamma \in \mathcal{HD}$, then it is clear that all the grafts of Γ are also elements of $\mathcal{HD}$. The set of all graft components, and hence the set of all grafts, can be ordered according to the ordering of the vertices of Γ; no two grafts will have the same underlying

vertex sets because diagrams with multiple edges between a pair of vertices are set to zero.

If $\Gamma \in \mathcal{HD}$, the grafts of Γ have an additional useful property which will allow us to build bundles over $\mathcal{H}$.

Proposition 4.14. *For $\Gamma \in \mathcal{HD}$, each graft of Γ has at most one segment vertex on each segment.*

Proof. First, we claim that for any pair of distinct free vertices v, v' in the same graft component $c(\widetilde{\Gamma})$, there exists a path of free edges between them. This is clear since each vertex of $\widetilde{\Gamma}$ which arises from a segment vertex of Γ is joined to precisely one other vertex in that component, so any path between v and v' in $c(\widetilde{\Gamma})$ can be shortened to avoid such vertices. This clearly descends to a path in c_Γ between v and v' consisting only of free edges.

Now suppose, on the contrary, that there is some graft component $c(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$ such that the associated graft $c(\Gamma)$ of Γ has two distinct segment vertices x and x' on a given segment.

Let $\alpha = \{e_i\}_{i=1}^k$ be any path of edges from x to x' in $c(\Gamma)$. Let $1 \leq j \leq k$ be such that $b(e_j)$ contains a segment vertex y on a segment different than the segment on which x, x' lie. Such a j must exist by definition of $\mathcal{HD}$. If $b(e_j) = \{y, v\}$ and $b(e_{j+1}) = \{y, v'\}$, then $v = v'$ implies y could be avoided by removing e_j, e_{j+1} from our path. Hence $v \neq v'$, and both are free vertices by Proposition 4.12. But our observation at the beginning of the proof shows there must exist a path between v and v' which avoids y. We can similarly eliminate any other segment vertex encountered along the way, producing a path between x and x' which does not pass through any other segment vertices. □

4.2.4. *Bundles of compactified configuration spaces from vertices and edges of a diagram*

We now describe the construction of bundles over $\mathcal{L}_m^n$ and $\mathcal{H}_m^n$ using the grafts of a diagram.

Proposition 4.15. *Let $\Gamma \in \mathcal{LD}$ be a diagram with i_j segment vertices on the jth segment, and let $c(\Gamma)$ be a graft of Γ with d_j segment vertices on the jth segment for all $j = 1$ to m. Let $\vec{i} = (i_1, \ldots, i_m)$. Then $c(\Gamma)$ gives rise to an evaluation map*

$$\mathrm{ev}_{c(\Gamma)} : \mathcal{L}_m^n \times C\left[\vec{i}; \coprod_{j=1}^m \mathbb{R}\right] \to C\left[\sum_j d_j, \mathbb{R}^n\right].$$

If $c_1(\Gamma), \ldots, c_k(\Gamma)$ are the grafts of Γ ordered as described above, and $c_l(\Gamma)$ has $d_{l,j}$ segment vertices on the jth segment for $l = 1$ to k, then we have an evaluation map

$$\mathrm{ev}_{\mathrm{gr}(\Gamma)} : \mathcal{L}_m^n \times C\left[\vec{i}; \coprod_{j=1}^m \mathbb{R}\right] \to \prod_{l=1}^k C\left[\sum_j d_{l,j}, \mathbb{R}^n\right], \tag{26}$$

where $\mathrm{ev}_{\mathrm{gr}(\Gamma)} = (\mathrm{ev}_{c_1(\Gamma)}, \ldots, \mathrm{ev}_{c_k(\Gamma)})$. *Moreover, if* $\Gamma \in \mathcal{HD}$, *then we have an evaluation map*

$$\mathrm{ev}_{\mathrm{gr}(\Gamma)} : \mathcal{H}^n_m \times C\left[\vec{i}; \coprod_{j=1}^{m} \mathbb{R}\right] \to \prod_{l=1}^{k} C\left[\sum_j d_{l,j}, \mathbb{R}^n\right]$$

whose restriction to $\mathcal{L}^n_m \times C[\vec{i}; \coprod_{j=1}^m \mathbb{R}]$ *is equal to the map in Eq.* (26), *and whose image in each factor lies in the open configuration space* $C(\sum_j d_{l,j}, \mathbb{R}^n)$.

Proof. This follows immediately from Proposition 4.14, since there is at most one segment vertex on each segment of a graft $c(\Gamma)$, and since homotopy links send points in distinct segments to distinct points, so that the codomain of the evaluation map is correctly identified. □

If $\Gamma \in \mathcal{LD}$, we now have a different evaluation maps associated with a link diagram, and this gives rise to a new way to build a bundle associated with a diagram.

Definition 4.16. Let $\Gamma \in \mathcal{LD}$ be a link diagram with grafts $c_1(\Gamma), \ldots, c_k(\Gamma)$ such that $c_l(\Gamma)$ has $d_{l,j}$ segment vertices on the jth segment and s_l free vertices for $l = 1$ to k. Let $\vec{d_l} = (d_{l,1}, \ldots, d_{l,m})$. Define

$$\oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)]$$

as the pullback of pr along $\mathrm{ev}_{gr(\Gamma)}$:

$$\begin{tikzcd} \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \arrow[r] \arrow[d] & \prod_{l=1}^k C[\sum_j d_{l,j} + s_l, \mathbb{R}^n] \arrow[d, "\mathrm{pr}"] \\ \mathcal{L}^n_m \times C[\vec{i}; \coprod_{j=1}^m \mathbb{R}] \arrow[r, "\mathrm{ev}_{\mathrm{gr}(\Gamma)}"] & \prod_{l=1}^k C[\sum_j d_{l,j}, \mathbb{R}^n]. \end{tikzcd} \tag{27}$$

Similarly, we define $\oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}^n_m, c_l(\Gamma)]$ when $\Gamma \in \mathcal{HD}$ and $\mathcal{H}^n_m$ replaces $\mathcal{L}^n_m$.

Remark 4.17. The notation here is meant to observe that given a collection of spaces and maps $X \to Y_i \leftarrow Z_i$ such that P_i is the pullback of this diagram for each index i, then the pullback of the evident diagram $X \to \prod_i Y_i \leftarrow \prod_i Z_i$ is the pullback of $\prod_i P_i$ along the diagonal map $\Delta : X \to \prod_i X$.

Proposition 4.18. *Let* $\Gamma \in \mathcal{LD}$ *be a link diagram with grafts* $c_1(\Gamma), \ldots, c_k(\Gamma)$ *such that* $c_l(\Gamma)$ *has* $d_{l,j}$ *segment vertices on the* j*th segment for* $l = 1$ *to* $k, j = 1$ *to* m. *Then the projection*

$$\pi_{\mathcal{L},\Gamma} : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \to \mathcal{L}^n_m$$

is a smooth fiber bundle whose fibers are smooth finite-dimensional manifolds with corners. Moreover, if $\Gamma \in \mathcal{HD}$, *then the projection*

$$\pi_{\mathcal{H},\Gamma} : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}^n_m, c_l(\Gamma)] \to \mathcal{H}^n_m$$

is also a smooth fiber bundle whose fibers are smooth finite-dimensional manifolds with corners, and

$$\begin{tikzcd} \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \arrow[r] \arrow[d, "\pi_{\mathcal{L},\Gamma}"'] & \oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}^n_m, c_l(\Gamma)] \arrow[d, "\pi_{\mathcal{H},\Gamma}"] \\ \mathcal{L}^n_m \arrow[r] & \mathcal{H}^n_m \end{tikzcd} \tag{28}$$

is a pullback square.

Proof. The projection $\pi_{\mathcal{L},\Gamma}$ is a smooth bundle for the same reasons that $\overline{\pi}_{\mathcal{L},\Gamma}$ in Proposition 4.6 is. For $\pi_{\mathcal{H},\Gamma}$, this is just an extension of the observation made in (25). (As mentioned for the bundle $\overline{\pi}_{\mathcal{L},\Gamma}$, it will sometimes be convenient to think of the bundle $\pi_{\mathcal{L},\Gamma}$ (respectively, $\pi_{\mathcal{H},\Gamma}$) as a compatible collection of bundles, one for each finite-dimensional manifold mapped into $\mathcal{L}^n_m$ (respectively, $\mathcal{H}^n_m$).) Lastly, the fact that the square (28) is a pullback follows directly from the definitions. □

We will denote the fibers of $\pi_{\mathcal{L},\Gamma}$ and $\pi_{\mathcal{H},\Gamma}$ over a link $L \in \mathcal{L}^n_m$ or a homotopy link $H \in \mathcal{H}^n_m$, respectively, by

$$\pi^{-1}_{\mathcal{L},\Gamma}(L) = \oplus_l C[\vec{d_l} + s_l;\, L, c_l(\Gamma)]$$

and

$$\pi^{-1}_{\mathcal{H},\Gamma}(H) = \oplus_l C[\vec{d_l} + s_l;\, H, c_l(\Gamma)].$$

Example 4.19. Consider the two different evaluation maps, one from Eq. (21) and the other from Eq. (26), for the diagram Γ from Fig. 16. For conciseness, we have omitted the compactification coordinates.

On the one hand, using Eq. (21), we have

$$\mathrm{ev}_\Gamma : \mathcal{L}^n_3 \times C[1, 2, 1; \mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R}] \to C[4, \mathbb{R}^n]$$

whose restriction to the interior is given by

$$(L, x, y_1, y_2, z) \mapsto (L(x), L(y_1), L(y_2), L(z)).$$

The image of this restriction lies in the subspace of all (w_1, w_2, w_3, w_4) where $w_1 \neq w_2, w_3, w_4$, and $w_2, w_3 \neq w_4$ of $(\mathbb{R}^n)^4$. We also have the projection map

$$\mathrm{pr} : C[5, \mathbb{R}^n] \to C[4, \mathbb{R}^n]$$

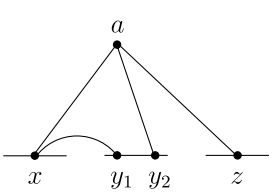

Fig. 16.

which on the interior sends $(w_1, w_2, w_3, w_4, w_5)$ to (w_1, w_2, w_3, w_4), so that the fibers of the bundle $\overline{\pi}_{\mathcal{L},\Gamma} : C[(1,2,1)+1; \mathcal{L}_3^n] \to \mathcal{L}_3^n$ are a subspace of $C[5, \mathbb{R}^n]$. The five configuration points correspond with the vertices of Γ, and we blow up all diagonals of $(\mathbb{R}^n)^5$. Note that the bundle obtained is exactly the same for any diagram with the same vertices as Γ.

On the other hand, Γ has two graft components, one of which is the diagram with a single chord from x to y_1, and the other of which is the "tripod" with free vertex a and edges between it and x, y_2, and z. Then Eq. (26) gives another evaluation map

$$\mathrm{ev}_{\mathrm{gr}(\Gamma)} : \mathcal{L}_3^n \times C[1,2,1; \mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R}] \to C[2, \mathbb{R}^n] \times C[3, \mathbb{R}^n]$$

given on the interior by

$$(L, x, y_1, y_2, z) \mapsto (L(x), L(y_1), L(x), L(y_2), L(z))$$

whose image in each factor lies in the open configuration space. To build the bundle, we use the product of two projection maps

$$C[2, \mathbb{R}^n] \times C[4, \mathbb{R}^n] \to C[2, \mathbb{R}^n] \times C[3, \mathbb{R}^n]$$

given by

$$(u_1, u_2, w_1, w_2, w_3, w_4) \mapsto (u_1, u_2, w_1, w_2, w_3)$$

to form a bundle

$$\pi_{\mathcal{L},\Gamma} : C[(1,1,0); \mathcal{L}_3^n] \oplus C[(1,1,1)+1; \mathcal{L}_3^n] \to \mathcal{L}_3^n.$$

The fibers of this bundle are isomorphic to a subspace of $(\mathbb{R}^n)^5$, namely the subspace of all tuples $(w_1, w_2, w_3, w_4, w_5) = (L(x), L(y_1), L(y_2), L(z), a)$, but $w_3 = w_5$ is now allowed and we do not blow up this diagonal. This is because there is no mixed edge between the free vertex a and the segment vertex y_1. We also do not blow up the locus $w_2 = w_3$. Thus, the fibers are a subspace of a (compactified) partial configuration space, because not all diagonals have been removed from $(\mathbb{R}^n)^5$.

In general, the difference between the pullback bundle based on vertices only and the one based on vertices and edges is precisely what we saw in the last example. In the latter, the configuration space is not compactified along all the diagonals but only along those that belong to the same graft component. Thus, if there is no edge between two vertices and they belong to different graft components, the corresponding configuration points can pass through each other without the direction of collision being recorded.

4.3. *Pullback of differential forms to new bundles of configuration spaces*

For the sake of concreteness, it is necessary to choose coordinates on our configuration spaces so that we may explicitly define the pullback of forms. As the interior

of configuration space is a subspace of a product of Euclidean spaces, it will suffice instead to consider coordinate systems on such spaces.

Given a finite ordered set S, we have a unique order-preserving isomorphism

$$\mathrm{pos}: S \to \{1, \ldots, |S|\}.$$

For a coordinate system $(x_1, \ldots, x_{|S|})$ on $(\mathbb{R}^n)^{|S|}$, this gives a natural way to associate $s \in S$ with the coordinate $x_{\mathrm{pos}(s)}$.

Suppose we have a category $\mathcal{C}$ whose objects are subsets of a fixed finite ordered set S and whose morphisms are inclusions. The association $T \mapsto (\mathbb{R}^n)^{|T|}$ is a contravariant functor from $\mathcal{C}$ to spaces, since an inclusion $T \to T'$ gives rise to the projection $p_i : (\mathbb{R}^n)^{|T'|} \to (\mathbb{R}^n)^{|T|}$ which forgets the coordinates associated with $T' - T$.

Now suppose we have a family of subsets $T_1, \ldots, T_k$ of S whose union is equal to S. We will let $\mathcal{C}$ be the category as above whose objects are S and all possible intersections of the T_i.

Consider the category of subsets of $\{1, \ldots, k\}$ with inclusions as morphisms. For each $R \subset \{1, \ldots, k\}$ we have the set $T_R := \bigcap_{i \in R} T_i$ (where we define $T_\emptyset := S$), and for each inclusion $R \to R'$ an inclusion $T_{R'} \to T_R$. Hence $R \mapsto T_R$ is a contravariant functor to $\mathcal{C}$, which can be thought of as a k-dimensional cube. Following this by the functor from $\mathcal{C}$ to spaces defined above gives a covariant functor $R \mapsto (\mathbb{R}^n)^{|T_R|}$. Since S is the union of all the T_i, we have that $\lim_{R \neq \emptyset} (\mathbb{R}^n)^{|T_R|} \cong (\mathbb{R}^n)^{|S|}$. The particular isomorphism we have in mind is the one which makes the following diagram commute:

$$\begin{tikzcd}
\lim_{R \neq \emptyset} (\mathbb{R}^n)^{|T_R|} \arrow[r] \arrow[dr] & (\mathbb{R}^n)^{|S|} \arrow[d, "(p_1, \ldots, p_k)"] \\
& \prod_{i=1}^k (\mathbb{R}^n)^{|T_i|}
\end{tikzcd}$$

The diagonal arrow is the natural inclusion of the limit into the product, and the top arrow is the isomorphism we spoke of above, and we use it to give coordinates on the limit. Given a diagram $\Gamma \in \mathcal{LD}$, the situation described above arises with $S = V(\Gamma)$ and T_i as the set of vertices of the ith graft (recall that the set of grafts is naturally ordered).

Definition 4.20. Let $\Gamma \in \mathcal{LD}$ be a diagram with i_j segment vertices on the jth segment and s free vertices. Let $e \in E(\Gamma)$, and suppose $b(e) = \{v, w\}$.

- If $v \neq w$, then if e is oriented from v to w (or if it is not oriented, then if $v < w$ in the ordering of the vertex set), define

$$\phi'_e : \prod_{l=1}^{k} C\left[\sum_j d_{l,j} + s_l, \mathbb{R}^n\right] \to S^{n-1}$$

as the map given on the interior by

$$\vec{x} \mapsto \frac{x_{\mathrm{pos}(w)} - x_{\mathrm{pos}(v)}}{|x_{\mathrm{pos}(w)} - x_{\mathrm{pos}(v)}|},$$

and define

$$\phi_e : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \to S^{n-1}$$

to be the pullback of ϕ'_e along the map $\oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \to \prod_{l=1}^k C[\sum_j d_{l,j} + s_l, \mathbb{R}^n]$.

- If $v = w$, then necessarily e joins a segment vertex with itself, and if it is oriented by the injection which sends $b(e) = \{v\}$ to 1 (or is not oriented at all),

$$\phi_e(\vec{x}, L) = D_z L(u)/|D_z L(u)|,$$

where z is the point in one of the strands such that $L(z) = x_{\mathrm{pos}(v)}$ and u is the positive unit tangent vector to the strand at z. If e is oriented by the injection sending $b(e) = \{v\}$ to -1, then

$$\phi_e(\vec{x}, L) = -D_z L(u)/|D_z L(u)|.$$

with z, u as above.

Note that $D_z L(u) \neq 0$ since L is an embedding; in the case of homotopy string links, which may not be embeddings, we do not have to worry about whether this is well-defined because loops cannot be present in diagrams in $\mathcal{HD}$.

Definition 4.21. Given $\Gamma \in \mathcal{LD}$ as above, define

$$\phi_\Gamma : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \to S^{(n-1)|E(\Gamma)|}$$

by

$$\phi_\Gamma = (\phi_{e_1}, \ldots, \phi_{e_{|E(\Gamma)|}}),$$

where $\mathrm{pos}(e_i) = i$ if the edge set is ordered, and otherwise order them according to the dictionary ordering on $\{b(e_i)\}$ (which can be imposed since diagrams with more than one edge joining a pair of vertices are set to zero).

Let $\mathrm{sym}_{S^{n-1}}$ be a smooth, unit volume top form on S^{n-1} which is symmetric (meaning its values on antipodal points are equal, though in Sec. 5.1, when we discuss the case of links in dimension 3, we will also require this form to be the unique rotation-invariant unit volume form) and let

$$\omega = \bigwedge_{|E(\Gamma)|} \mathrm{sym}_{S^{n-1}}.$$

Finally define the pullback form

$$\alpha_\Gamma = (\phi_\Gamma)^* \omega \in \Omega^{(n-1)(|E(\Gamma)|)}(\oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)]).$$

Notice that nothing changes in the case of homotopy links. For a diagram $\Gamma \in \mathcal{HD}$, we again use edges (but there are no longer any loops) to pull back a product of forms ω from $S^{(n-1)|E(\Gamma)|}$ to the space $\oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}^n_m, c_l(\Gamma)]$, although we will write $\alpha^{\mathcal{H}}_\Gamma$ for the pullback form when $\Gamma \in \mathcal{HD}$.

Observe also that the same definitions are valid for the bundle $C[\vec{i} + s; \mathcal{L}^n_m, \Gamma]$ considered in earlier literature on the subject. Namely, we have a map

$$\overline{\phi}_\Gamma : C[\vec{\imath} + s; \mathcal{L}^n_m, \Gamma] \to S^{(n-1)|E(\Gamma)|}$$

dictated by the edges of Γ, and this can be used for pulling back a product of volume forms to give a form $\overline{\alpha}_\Gamma = (\overline{\phi}_\Gamma)^*\omega$. This case was considered in [31, Sec. 3.2].

4.4. *Configuration space integrals of string links and homotopy string links*

We are finally ready to produce forms on spaces of links and homotopy links. Namely, the form α_Γ can be pushed forward, or integrated along the fiber of the bundle

$$\pi_{\mathcal{L},\Gamma} : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)] \to \mathcal{L}^n_m$$

to produce a form $(\pi_{\mathcal{L},\Gamma})_*\alpha_\Gamma$, or, as we will usually denote it, a form

$$(I_{\mathcal{L}})_\Gamma \in \Omega^{|\Gamma|}(\mathcal{L}^n_m).$$

The value of this form on a link $L \in \mathcal{L}^n_m$ is thus

$$(I_{\mathcal{L}})_\Gamma(L) = \int_{\pi^{-1}_{\mathcal{L},\Gamma}(L) = \oplus_l C[\vec{d_l} + s_l;\, L, c_l(\Gamma)]} \alpha_\Gamma.$$

The degree $|\Gamma| := (n-1)|E(\Gamma)| - n|V(\Gamma)_{\text{free}}| - |V(\Gamma)_{\text{seg}}|$ of $(I_{\mathcal{L}})_\Gamma$ is the difference of the degree of α_Γ and the dimension of the fiber $\pi^{-1}_{\mathcal{L},\Gamma}(L)$. Recall that this quantity is also equal to

$$k(n-3) + d,$$

where $d = \deg(\Gamma)$ and $k = \operatorname{ord}(\Gamma)$, so that we have constructed a map

$$I_{\mathcal{L}} : \mathcal{LD}^d_k \to \Omega^{k(n-3)+d}(\mathcal{L}^n_m). \tag{29}$$

For a diagram $\Gamma \in \mathcal{HD}$, we integrate the associated form $\alpha^{\mathcal{H}}_\Gamma$ along the bundle

$$\pi_{\mathcal{H},\Gamma} : \oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}^n_m, c_l(\Gamma)] \to \mathcal{H}^n_m.$$

This gives a form

$$(I_{\mathcal{H}})_\Gamma \in \Omega^{|\Gamma|}(\mathcal{H}^n_m)$$

whose value on a homotopy link $H \in \mathcal{H}^n_m$ is

$$(I_{\mathcal{H}})_\Gamma(H) = \int_{\pi^{-1}_{\mathcal{H},\Gamma}(H) = \oplus_l C[\vec{d_l} + s_l;\, H, c_l(\Gamma)]} \alpha^{\mathcal{H}}_\Gamma.$$

Again rewriting the degree of the form, we thus have a map

$$I_{\mathcal{H}} : \mathcal{HD}_k^d \to \Omega^{k(n-3)+d}(\mathcal{H}_m^n). \tag{30}$$

Remark 4.22. Thinking of the bundle as a collection of compatible bundles (as mentioned around Proposition 4.6) makes clear that this construction produces differential forms on $\mathcal{L}_m^n$ and $\mathcal{H}_m^n$ in the sense described at the end of Sec. 2.2. In fact, if we replaced the link space by any finite-dimensional manifold M (or just an open subset of Euclidean space) parametrizing a family of links, then fiberwise integration certainly produces a differential form on M. It is also clear that for another manifold M' mapped into $\mathcal{L}_m^n$ through the map $\psi : M' \to M$, the form on M' is the pullback via ψ of the form on M.

Remark 4.23. It is immediate from the definition that maps $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are also compatible with the inclusion

$$\mathcal{L}_m^n \hookrightarrow \mathcal{H}_m^n,$$

that is, we have a commutative diagram

$$\begin{tikzcd} \mathcal{HD} \arrow[r, hook] \arrow[d, "I_{\mathcal{H}}"'] & \mathcal{LD} \arrow[d, "I_{\mathcal{L}}"] \\ \Omega^*(\mathcal{H}_m^n) \arrow[r] & \Omega^*(\mathcal{L}_m^n) \end{tikzcd}$$

This is precisely what we were after when we refined the definition of the bundles we integrate over.

Now note that again nothing changes for the case of the pullback bundle defined without consideration of the grafts. Namely, the construction of $(\pi_{\mathcal{L},\Gamma})_*\alpha_\Gamma$ goes through exactly the same way to give a form $(\overline{\pi}_{\mathcal{L},\Gamma})_*\overline{\alpha}_\Gamma$ by pushing forward the form $\overline{\alpha}_\Gamma$ along the map

$$\overline{\pi}_{\mathcal{L},\Gamma} : C[\vec{\imath} + s; \mathcal{L}_m^n, \Gamma] \to \mathcal{L}_m^n$$

from Proposition 4.6. We now want to show that the forms we obtain by integrating along this bundle are the same as the forms we obtain by integrating along

$$\pi_{\mathcal{L},\Gamma} : \oplus_l C[\vec{d}_l + s_l;\ \mathcal{L}_m^n, c_l(\Gamma)] \to \mathcal{L}_m^n$$

are the same as in the case of integration along the bundle

$$\overline{\pi}_{\mathcal{L},\Gamma} : C[\vec{\imath} + s; \mathcal{L}_m^n, \Gamma] \to \mathcal{L}_m^n.$$

This will finally show that our way of setting up configuration space integrals for links is indeed a refinement of the way that has been considered in literature thus far.

Proposition 4.24. *For any* $\Gamma \in \mathcal{LD}$, $(\pi_{\mathcal{L},\Gamma})_*\alpha_\Gamma = (\overline{\pi}_{\mathcal{L},\Gamma})_*\overline{\alpha}_\Gamma$.

Proof. The map between fibers is the inclusion of an open dense set. The two fibers are the same on the biggest stratum, namely the open configuration space. They differ in that $\overline{\pi}^{-1}_{\mathcal{L},\Gamma}(L)$ has more diagonals of $\mathbb{R}^{n|V(\Gamma)|}$ removed and compactified. Thus, the difference between the two is at least of codimension-one and so the integrals are equal. □

We next give a few examples of these configuration space integrals.

Example 4.25 (Diagrams with no free vertices). One special case is that of diagrams with no free vertices, i.e. those that only contain chords and loops. In that case, the construction simplifies since there are no pullback constructions as in Definition 4.16, and the bundles constructed are trivial. For example, if $\Gamma \in \mathcal{LD}$ is the diagram from Fig. 17 (where we have omitted the edge orientations and labels for simplicity), then the map ϕ_Γ is a composition

$$\phi_\Gamma : \mathcal{L}^n_3 \times C[3,1,2;\mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R}] \xrightarrow{\mathrm{ev}_\Gamma} C[2,\mathbb{R}^n]^4 \times C[1,\mathbb{R}^n] \to (S^{(n-1)})^5.$$

After pulling back the product of five (antipodally) symmetric top forms from $(S^{(n-1)})^5$, the integration takes place along the trivial bundle

$$\pi_{\mathcal{L},\Gamma} : \mathcal{L}^n_3 \times C[3,1,2;\mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R}] \to \mathcal{L}^n_3.$$

Example 4.26 (Linking number). Another special case, and in fact the case that motivated Bott and Taubes to define configuration space integrals for knots in [5], is that of the linking number of a two-component link in $\mathbb{R}^3$. Namely, suppose Γ is the diagram with a single chord between segments i and j and no free vertices or segment vertices on other segments, as in Fig. 18.

Then the integration described above recovers the classical Gauss integral computing the linking number of strands i and j of a link or a homotopy link L, which

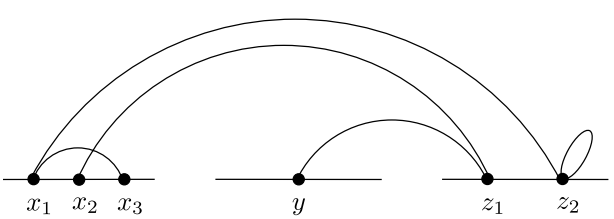

Fig. 17.

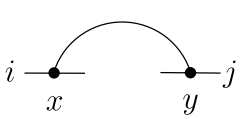

Fig. 18.

we will denote by $\mathrm{lk}(L_i, L_j)$. In short,

$$\mathrm{lk}(L_i, L_j) = (I_{\mathcal{H}})_\Gamma(L) = (I_{\mathcal{L}})_\Gamma(L) = \int_{C[1,1;\mathbb{R}\sqcup\mathbb{R}]} \left(\frac{L(x)-L(y)}{|L(x)-L(y)|}\right)^* \mathrm{sym}_{S^2},$$

where the compactification $C[1,1;\mathbb{R}\sqcup\mathbb{R}]$ is an octagonal disk (see [15, Sec. 1.2] for details).

To see how shuffle products of integrals give products of linking numbers, see Example 4.30.

Example 4.27 (Homotopy links with one strand). Consider the case of $\mathcal{H}_1^n, n \geq 3$. Now the only diagram in $\mathcal{HD}$ is the empty diagram, and so the integration does not produce any forms in this case. This is of course consistent with the fact that $\mathcal{H}_1^n$ is a contractible space (Corollary 2.4).

4.5. *Integration is a map of differential graded algebras*

The goal of this section is to prove Theorem 4.33, which says the map that associates fiberwise integrals to diagrams is a map of differential graded algebras. This theorem will follow from Propositions 4.28–4.31. Most of the statements follow easily from the case of knots considered in [7, 8], but for completeness and the convenience of the reader, we give fairly complete outlines of their proofs. We elaborate on the fact that $I_{\mathcal{L}}$ is a map of algebras; this result is stated in [8] but without justification. In addition, we also observe that the same proofs apply for the case of the map $I_{\mathcal{H}}$, and that in fact some of the results now even work for $n = 3$.

We begin with the following proposition.

Proposition 4.28. *For $n \geq 3$ and $m \geq 1, I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are well-defined homomorphisms.*

Proof. We check that integration is compatible with the relations from Definition 3.11. For the first condition, if Γ has a double edge, then ϕ_Γ factors through a product with one fewer sphere, since one direction is repeated:

$$\begin{tikzcd}
\oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}_m^n, c_l(\Gamma)] \arrow[rr, "\phi_\Gamma"] \arrow[dr] & & S^{(n-1)|E(\Gamma)|} \\
& S^{(n-1)(|E(\Gamma)|-1)} \arrow[ur] &
\end{tikzcd}$$

Then the pullback of ω via ϕ_Γ is the same as the pullback through the factorization. However, the dimension of ω is greater than $(n-1)(|E(\Gamma)|-1)$ and so the pullback is zero. The same argument holds when $\oplus_l C[\vec{d_l} + s_l;\, \mathcal{L}_m^n, c_l(\Gamma)]$ is replaced by $\oplus_l C[\vec{d_l} + s_l;\, \mathcal{H}_m^n, c_l(\Gamma)]$.

The other two conditions in Definition 3.11 are in fact designed for compatibility with the integration. Namely, if n is even or odd, then switching two configuration

points on the link (i.e. switching two copies of $\mathbb{R}$) gives $\oplus_l C[\vec{d_l} + s_l; \mathcal{L}^n_m, c_l(\Gamma)]$ and $\oplus_l C[\vec{d_l} + s_l; \mathcal{H}^n_m, c_l(\Gamma)]$ different orientations and produces an integral with a different sign. A similar situation occurs if two free configuration points are switched and n is odd, and if two maps are switched in the product ϕ_Γ and n is odd (this corresponds to switching the order of edges). The latter case introduces a sign because the effect is that of transposition of two even-dimensional forms. Again, a minus sign is introduced in the integral. Thus, $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are well-defined and they are homomorphisms since pullback of forms and integration are linear. □

Proposition 4.29. *For $n \geq 3$ and $m \geq 1, I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are maps of graded algebras.*

Proof. Recall that we can consider $\mathcal{LD}^*_*$ and $\mathcal{HD}^*_*$ as differential graded algebras with a single grading given by $|\Gamma|$. Since $I_{\mathcal{L}}(\Gamma)$ (or $I_{\mathcal{H}}(\Gamma)$) is a form of degree $|\Gamma|, I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ preserve this grading.

Thus, it remains to check that the shuffle product of diagrams from Definition 3.21 corresponds precisely to the wedge product of forms which gives the deRham complex the structure of an algebra. That is, we must check that

$$(I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2} = (I_{\mathcal{L}})_{\Gamma_1} \wedge (I_{\mathcal{L}})_{\Gamma_2} \quad \text{and} \quad (I_{\mathcal{H}})_{\Gamma_1 \bullet \Gamma_2} = (I_{\mathcal{H}})_{\Gamma_1} \wedge (I_{\mathcal{H}})_{\Gamma_2}. \tag{31}$$

This statement is a direct generalization of the same statement for long knots [8, Proposition 5.3]. Since that result is provided without much explanation, we elaborate on (31) a bit here.

Recall that one way to think about the wedge product is as follows:

Given a k-form α and an l-form β, the wedge product is a multilinear $(k+l)$-form whose value on the variables $x_1, \ldots, x_{k+l}$ is

$$\begin{aligned}\alpha \wedge \beta(x_1, \ldots, x_{k+l}) = \sum_{\sigma \in \mathrm{Shuffle}(k,l)} & \mathrm{sign}(\sigma)\alpha(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}) \\ & \times \beta(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(k+l)}),\end{aligned}$$

where Shuffle is the subset of the permutations of $\{1, \ldots, k+l\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l)$.

Thus, given diagrams Γ_1 and Γ_2, each shuffle $v_{\sigma(1)}, \ldots, v_{\sigma(k+l)}$ of the segment vertices on one segment corresponds to configurations on a strand of a link appearing in that order. In other words, the integration takes place over a "piece" of $\mathbb{R}^{k+l}$ determined by $x_{\sigma(1)} < \cdots < x_{\sigma(k+l)}$ (plus as many copies of $\mathbb{R}^n$ as there are free vertices in both diagrams, since they are free to move anywhere). Adding the integrals over all shuffles, we get $(I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2}$, and in this sum, integration thus takes place over all pieces of $\mathbb{R}^{k+l}$.[b] The integrals agree on the boundary, so that this sum can be represented by a single integral, taken over $\mathbb{R}^{k+l}$ (again plus some copies of $\mathbb{R}^n$). But this integral is a product of integrals by Fubini's Theorem, one taken over $\mathbb{R}^k$ and one over $\mathbb{R}^l$ (plus as many copies of $\mathbb{R}^n$ in each as there are free vertices

[b]This is much like what happens in the Eilenberg–Zilber map.

in the two diagrams whose shuffle product was taken). This product of integrals is precisely $(I_{\mathcal{L}})_{\Gamma_1} \wedge (I_{\mathcal{L}})_{\Gamma_2}$. The same is true when $I_{\mathcal{L}}$ is replaced by $I_{\mathcal{H}}$. □

An example of the argument given above is the following.

Example 4.30. Recalling Example 4.26, we now also see from Proposition 4.29 how shuffle products of diagrams, each with one chord between different strands, correspond to the powers and products of linking numbers. For example, if Γ_1 and Γ_2 are as in Fig. 19, then their shuffle product is given in Fig. 20.

The corresponding sum of integrals is the following (with explanations below):

$$
\begin{aligned}
&(I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2}(L) \\
&\quad = (I_{\mathcal{H}})_{\Gamma_1 \bullet \Gamma_2}(L) \\
&\quad = \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_1 \le x_2} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&\qquad + \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_1 \le x_2} \left(\frac{L(x_1)-L(z)}{|L(x_1)-L(z)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(y)}{|L(x_2)-L(y)|}\right)^* \mathrm{sym}_{S^2} \\
&\quad \overset{\text{(i)}}{=} \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_1 \le x_2} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&\qquad + \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_2 \le x_1} \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2}
\end{aligned}
$$

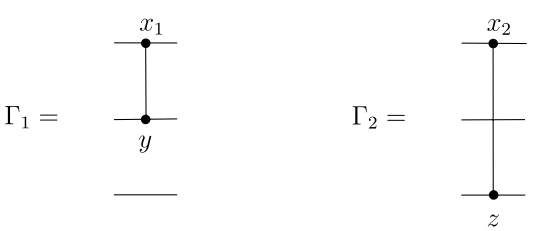

Fig. 19.

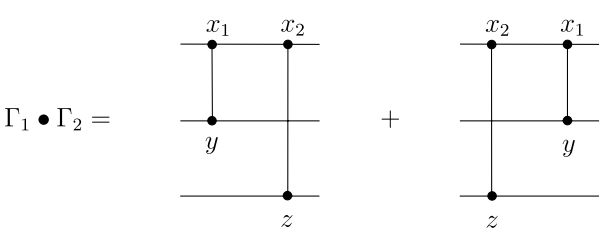

Fig. 20.

$$
\begin{aligned}
&\overset{\text{(ii)}}{=} \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_1\le x_2} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&\quad + \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_2\le x_1} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&\overset{\text{(iii)}}{=} \int_{C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:(x_1,x_2)\in\mathbb{R}^2} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \wedge \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&\overset{\text{(iv)}}{=} \int_{(x_1,y)\in C[1,1;\mathbb{R}\sqcup\mathbb{R}]} \left(\frac{L(x_1)-L(y)}{|L(x_1)-L(y)|}\right)^* \mathrm{sym}_{S^2} \\
&\quad \cdot \int_{(x_2,z)\in C[1,1;\mathbb{R}\sqcup\mathbb{R}]} \left(\frac{L(x_2)-L(z)}{|L(x_2)-L(z)|}\right)^* \mathrm{sym}_{S^2} \\
&= \mathrm{lk}(L_1,L_2)\cdot \mathrm{lk}(L_1,L_3).
\end{aligned}
$$

The subscript $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:x_1\le x_2$ indicates integration over the component of $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]$ whose interior consists of points $(-\infty < x_1 < x_2 < \infty, y\in\mathbb{R}, z\in\mathbb{R})$.

Equality (i) comes from just switching the labels x_1 and x_2. Equality (ii) holds because switching the order of the maps, and hence pullbacks, does not matter ($n=3$ is odd here).

In equality (iii), the subscript $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]:(x_1,x_2)\in\mathbb{R}^2$ denotes the space obtained by gluing the two components of $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]$ along the boundary face where x_1 has collided with x_2. (This space can also be constructed in a similar way to $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]$, but without blowing up the diagonal $x_1=x_2$.) Here, we use that the two integrals on the previous line agree on this boundary face. The diagram representing this boundary in both cases is the one in Fig. 21. This boundary faces indeed has opposite orientations in the two components of $C[2,1,1;\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}]$.

In equality (iv), we use that the maps used to pull back sym_{S^2} factor through a configuration space where all the faces at infinity except those corresponding to $\{x_1,y,\infty\}$ and $\{x_2,z,\infty\}$ are collapsed to points (i.e. a configuration space obtained by blowing up only those two diagonals). This space is the product $(C[1,1;\mathbb{R}\sqcup\mathbb{R}])^2$,

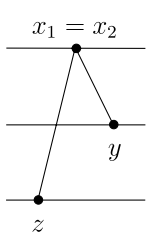

Fig. 21.

to which we apply Fubini's theorem. Lastly, note that in the expression following equality (iv), we get the ordinary product of integrals, rather than a wedge product, since the forms we obtain are 0-forms, i.e. functions on $\mathcal{L}_3^3$ (or $\mathcal{H}_3^3$), and the wedge product in that case is the usual product.

Proposition 4.31. *For $n \geq 4$ and $m \geq 1, I_{\mathcal{L}}$ is a map of differential complexes. For $n \geq 3$ and $m \geq 1$, the same is true for $I_{\mathcal{H}}$.*

Remark 4.32 (Erratum to [31]). In the case of string links, this proposition reduces to the statement of [31, Theorem 3.7]. However, with the definition of string links used in that paper, it is unclear how to compactify the configuration space as points on the string link approach infinity. While our present definition of string links fixes that issue, the proof of "vanishing along faces at infinity" in [31] is still incomplete. Thus, the proof of this proposition provides an erratum to [31]. This will justify all the statements in that paper which depend on the vanishing of the integrals along faces at infinity.

Proof of Proposition 4.31. The proof is very similar to the proof of the corresponding result for closed knots and $n \geq 4$, established in the Appendix of [7]. In short, Stokes' Theorem implies that

$$d((\pi_{\mathcal{L},\Gamma})_* \alpha_\Gamma) = (\pi_{\mathcal{L},\Gamma})_* d\alpha_\Gamma + (\partial \pi_{\mathcal{L},\Gamma})_* \alpha_\Gamma. \tag{32}$$

Since in our case α_Γ is the pullback of a closed form (namely the product of volume forms on the sphere), $d\alpha_\Gamma = 0$. Thus, the right-hand side is just $(\partial \pi_{\mathcal{L},\Gamma})_* \alpha_\Gamma$, where this term denotes the sum of integrals along all codimension-one faces of $\oplus_l C[\vec{d_l} + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$. The faces given by two points colliding, called *principal*, correspond to contractions of edges in $\mathcal{LD}$. To get a map of complexes, therefore, it remains to show the vanishing of the restriction of the integral to all other faces. Recalling the discussion following Definition 4.2, such faces are characterized by more than two points coming together at the same time or one or more points escaping to infinity. The former are called *hidden faces*, and the latter are called *faces at infinity.*

The vanishing arguments depend on the various cases. In some cases, there is an involution of the face which either preserves its orientation and negates the form to be integrated, or reverses its orientation and preserves the form; thus the integral vanishes (see, for example, [30, Lemmas 4.5 and 4.6]). The remaining cases depend on dimension-counting. A representative dimension-counting argument is given in the beginning of the proof of Proposition 4.28.

For the case of closed knots and $n \geq 4$, the details of these vanishing arguments can be found in [7, 30]. The authors of [7] argue by partitioning the faces into three types: Type I corresponds to collisions of free vertices away from ∞, Type II corresponds to collisions of free vertices with ∞, and Type III corresponds to collisions of both free and segment vertices away from ∞. The generalization of their arguments to closed links and $n \geq 4$ is immediate. To generalize to string links, including long

knots, one just has to address faces where $r+s$ points approach infinity, $r \geq 1$ of which are on the link. We call his a Type IV face.

This Type IV face is similar to a Type II face, where s points, none of which are constrained to the link, approach infinity. In a Type II face, the collision of s points with ∞ is described by a screen, which is a point in the space $C(s+1, T_\infty S^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+)$. (Here, $\mathbb{R}^n \rtimes \mathbb{R}_+$ is the group of translations and oriented scalings of $T_\infty S^n$.) By fixing the last point at ∞, we can write this space as $C(s, T_\infty S^n \backslash \{0\})/\mathbb{R}_+$. For the Type IV face, where $r+s$ points go to infinity with the first r of them on the link L, we replace $C(r+s, T_\infty S^n \backslash \{0\})/\mathbb{R}_+$ by the subspace where the r points lie on appropriate components of $T_\infty L$. The dimension of this "screen-space" is $r+ns-1$.

Alternatively, we can describe the screen from the viewpoint of the origin rather than ∞. In this description, the screen is a point in $C(r+s+1, \mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+)$. Here, the last point corresponds to the collection of points that have not escaped to infinity. Heuristically, if Γ' is the subgraph of vertices that escape to infinity, then the complement of Γ' is collapsed to a point in this description. By translating this point to the origin, this space is the same as $C(r+s, \mathbb{R}^n \backslash \{0\})$. Since the first r points are on the link L, the screen lies in the subspace where the first r points are constrained to appropriate rays through the origin, corresponding to the linear behavior of L toward ∞.

For every such face at infinity $\mathfrak{S}$, consider the map $\mathfrak{S} \to (S^{n-1})^{|E(\Gamma)|}$. (The description from the viewpoint of the origin above makes it particularly easy to see what the map is for the factors of S^{n-1} indexed by edges joining vertices in Γ' to vertices outside Γ'.) This map can be factored through a product of two maps, one of which is from the (finite-dimensional) screen-space to $(S^{n-1})^{|E(\Gamma')|}$, where $\Gamma' \subset \Gamma$ consists of the vertices which have gone to infinity. As in [7, Lemmas A.7–A.9], we first reduce to the case where every free vertex in Γ' has valence ≥ 3, and every segment vertex in Γ' has valence ≥ 1:

Indeed, if v is any vertex which is 0-valent in Γ' or a free vertex which is 1-valent in Γ', then v is joined by some edge e to a vertex outside of Γ'. Then the map from $\mathfrak{S} \to (S^{n-1})^{|E(\Gamma)|}$ is constant in the S^{n-1} factor determined by e. Thus, the image of this map has codimension $\geq n-1$. So as in the beginning of the proof of Proposition 4.28, the form to be integrated is pulled back through a lower-dimensional space and hence vanishes. Finally, if there is a vertex which is bivalent in Γ', then the involution of the screen-space (due to Kontsevich) guarantees the vanishing of the integral along $\mathfrak{S}$ (see [7, Lemma A.9]).

So we may now suppose Γ' is "at least unitrivalent". We claim the dimension of the screen-space $r+ns-1$ is less than $(n-1)|E(\Gamma')|$. In fact, we have

$$(n-1)|E(\Gamma')| - (r+ns-1) \geq (n-1)\frac{r+3s}{2} - (r+ns-1) \tag{33}$$

$$= \frac{(n-3)(r+s)}{2} + 1 \tag{34}$$

$$= \frac{(n-3)(r+s-2)}{2} + n - 2 \tag{35}$$

$$\geq 1 \tag{36}$$

since $n \geq 3$ and, by our assumptions on the valences in $\Gamma', r+s \geq 2$. So again, the pulled-back form to be integrated factors through a lower-dimensional space and hence vanishes. This proves the first statement of the proposition.

The same arguments of the Appendix of [7] together with our addendum above for string links show that $I_{\mathcal{H}}$ is a chain map. (Alternatively, for $n \geq 4$, we can use that $\mathcal{HD}$ is a subcomplex of $\mathcal{LD}$, so we get a chain map $I_{\mathcal{H}}$ by restricting $I_{\mathcal{L}}$ to $\mathcal{HD}$.) Moreover, these arguments apply when $n = 3$ to every face except Type III face where the subgraph Γ' corresponding to the collided vertices is "at least unitrivalent". In defect zero, such a face must be the hidden face where all the configuration points come together (away from ∞), i.e. the so-called *anomalous face*. This face will be discussed further in Sec. 5.1. Note that a collision of all configuration points can only happen if all the segment vertices in a diagram $\Gamma \in \mathcal{LD}$ are concentrated on one segment (see Remark 5.1). However, it is immediate from the definition of $\mathcal{HD}$ that no $\Gamma \in \mathcal{HD}$ can have all its segment vertices on one segment, unless Γ is the empty diagram. Therefore, one never encounters an anomalous face in the case of $I_{\mathcal{H}}$. Thus, the arguments above show that we also get a chain map in the case of homotopy links for $n = 3$ in defect zero (which is also main degree zero), even though $I_{\mathcal{L}}$ is not known to be a chain map for $n = 3$. □

Let $I^0_{\mathcal{L}}$ and $I^0_{\mathcal{H}}$ denote the restrictions of $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ to $\mathcal{LD}^0_*$ and $\mathcal{HD}^0_*$. For $n \geq 4$, one can show that $I^0_{\mathcal{L}}$ induces an injective map in cohomology. The proof of this fact proceeds exactly as in the case of closed knots in [7], to which we refer the reader for details. For $I^0_{\mathcal{H}}$ consider the following diagram:

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{HD}^*_k) \arrow[r, hook] \arrow[d, "I^0_{\mathcal{H}}"'] & \mathrm{H}^0(\mathcal{LD}^*_k) \arrow[d, hook, "I^0_{\mathcal{L}}"] \\ \mathrm{H}^{k(n-3)}(\mathcal{H}^n_m) \arrow[r] & \mathrm{H}^{k(n-3)}(\mathcal{L}^n_m) \end{tikzcd} \tag{37}$$

The top horizontal map is an injection because the degree zero cohomologies are just subspaces of $\mathcal{HD}^0_k$ and $\mathcal{LD}^0_k$; hence this arrow is just a restriction of the inclusion $\mathcal{HD}^0_k \hookrightarrow \mathcal{LD}^0_k$. We just alluded to the proof that the right-hand vertical map is an inclusion. The bottom horizontal map is induced by the inclusion $\mathcal{L}^n_m \hookrightarrow \mathcal{H}^n_m$. From the definitions of $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$, we see that this square commutes. Thus, the left vertical map is an injection. Putting this together with the previous three propositions, we have the following.

Theorem 4.33. *For $n \geq 4$ and $m \geq 1$, the integration map*

$$I_{\mathcal{L}} : \mathcal{LD}_k^d \to \Omega^{k(n-3)+d}(\mathcal{L}_m^n),$$
$$\Gamma \mapsto \left(L \mapsto (I_{\mathcal{L}})_\Gamma(L) = \int_{\pi_{\mathcal{L},\Gamma}^{-1}(L) = \oplus_l C[\vec{d}_l + s_l;\, L, c_l(\Gamma)]} \alpha_\Gamma \right) \tag{38}$$

induces a morphism of differential graded algebras. We recall here that the grading $|\Gamma|(= k(n-3)+d)$ (together with the differential δ and the shuffle product) makes the left-hand side a differential graded algebra, while the right-hand side is just the de Rham complex of $\mathcal{L}_m^n$.

For $n \geq 3$ and $m \geq 1$, the same is true of the map

$$I_{\mathcal{H}} : \mathcal{HD}_k^d \to \Omega^{k(n-3)+d}(\mathcal{H}_m^n),$$
$$\Gamma \mapsto \left(H \mapsto (I_{\mathcal{H}})_\Gamma(H) = \int_{\pi_{\mathcal{H},\Gamma}^{-1}(H) = \oplus_l C[\vec{d}_l + s_l;\, H, c_l(\Gamma)]} \alpha_\Gamma^{\mathcal{H}} \right). \tag{39}$$

For $n \geq 4$ and $d = 0$, the maps induced in cohomology by both of these maps are injective.

Remark 4.34. The results proven in Sec. 5 will imply that for $n = 3, I_{\mathcal{H}}^0$ induces an injection in cohomology.

Remark 4.35. Conjecturally, the map $I_{\mathcal{L}}$ is a quasi-isomorphism. This is likely since it is known that $\mathcal{LD}$ and $\mathcal{L}_m^n$ have isomorphic cohomology.

Theorem 4.36. *For $n \geq 5$, changing the form $\mathrm{sym}_{S^{n-1}}$ to another (antipodally) symmetric volume form does not affect the map $I_{\mathcal{L}}$ in cohomology. For $n \geq 4$, such a change of form does not affect the map $I_{\mathcal{H}}$ in cohomology.*

Proof. The idea of the proof is the same as in [7, Proposition 4.5, Sec. A.4] (see also [29, Sec. 4.2]). If α_1, α_2 are two forms on the same total space coming from two different volume forms, then their difference is an exact form $d\beta$. We want to show that the fiberwise integral of $d\beta$ is exact. By Eq. (32) (Stokes' Theorem), this integral is the difference of an exact form and the integral along the boundary of the fiber of β. Thus, it suffices to show that this integral along the boundary vanishes. As before, we can do this either by involutions of boundary faces or by dimension-counting arguments. However, since β is a *primitive* for $\alpha_1 - \alpha_2$, our dimension-counting arguments must show that the image of a boundary face of the total space in the product of spheres has codimension at least *two* (rather than one).

The proof of [7, Proposition 4.5, Sec. A.4] for knots treats the case of Type I, II, and III faces for $n \geq 5$. So to prove the theorem statement for $I_{\mathcal{L}}$ we just need to treat the Type IV face. For the faces $\mathfrak{S}$ where the corresponding subgraph Γ' is "less than unitrivalent", we saw that either the image of the face $\mathfrak{S}$ in the product

of spheres has codimension $n-1$ (≥ 2), or $\mathfrak{S}$ has an involution that guarantees the vanishing of the integral. For the case where Γ' is "at least unitrivalent", our calculation in (33) shows that for $n \geq 4$, the quantity $(n-1)|E(\Gamma')| - (r+ns-1)$ is ≥ 2, and hence that the codimension of the image of this Type IV face in the product of spheres is ≥ 2. This proves the theorem for $n \geq 5$.

For the statement regarding $I_{\mathcal{H}}$ when $n=4$, we note that the argument fails for $I_{\mathcal{L}}$ and $n=4$ because of the Type III face. In the case of ordinary (long) knots/links, this face is the pullback via the unit derivative map of a bundle over S^{n-1}. However, for homotopy links, because of our grafts, such a codimension-one face can involve only $r=1$ point on the link. In this case, the description of this face is the same as a Type I face (where only free points collide). A Type I face does not involve tangential data and can be dealt with by a dimension-counting argument. Thus, we get the desired statement for $I_{\mathcal{H}}$ when $n=4$. The reader may consult the Appendix of [7] for further details. □

We cannot necessarily extend the result concerning $I_{\mathcal{H}}$ to $n=3$ because in that case the image of the Type IV face in the product of spheres may have codimension-one.

5. Configuration Space Integrals and Finite Type Invariants of Homotopy String Links

In this section, we focus on classical homotopy links, so $n=3$, and we want to see what invariants, i.e. forms in degree zero, one obtains through our integration. It turns out that what appears are precisely *finite type invariants* of homotopy links, and that is the main result of this section. One way of saying this is that the vector space of weight systems $\mathcal{HW}_k$ from Sec. 3.4 corresponds precisely to $\mathbb{R}$-valued finite type k invariants of homotopy links via configuration space integrals. That the two are isomorphic is known [4], but we exhibit this isomorphism explicitly using configuration space integrals. For links, this statement appeared in [31, Sec. 4] and is for convenience restated below as Theorem 5.6. The bulk of this section is devoted to proving the same statement for homotopy links (Theorem 5.8). However, since the proofs are essentially identical for links and homotopy links, and since we supply most of the details here, this section can be thought of as also giving the proof of Theorem 5.6. See Remark 5.12 for more details.

One important difference between links and homotopy links in this section is that one no longer has to worry about anomalous faces in the case of homotopy links (see Remark 5.1). Looking at Eq. (29), we see that it is precisely diagrams in defect zero that give degree zero forms, so this is why we considered them in Sec. 3.4; the reader may find it helpful to review that section before proceeding with this one. Since the rest of the paper only deals with $n=3$, one may now safely confuse diagrams of defect zero with those of main degree zero, since the two coincide for $n=3$.

5.1. *The anomalous correction*

As mentioned in the proof of Proposition 4.31, the map $I_{\mathcal{L}}$ is not a chain map for $n = 3$. Recall that, to prove that $I_{\mathcal{L}}$ commutes with the differential, we have to check the vanishing of certain integrals along the hidden faces or faces at infinity of $\oplus_l C[\vec{d_l} + s_l; \mathcal{L}^n_m, c_l(\Gamma)]$. That is, for $\Gamma \in \mathcal{LD}^d_k$, Stokes' Theorem implies that

$$(d(I_{\mathcal{L}}(\Gamma)))(L) = \int_{\partial(\pi^{-1}_{\mathcal{L},\Gamma}(L))} \alpha_\Gamma,$$

and if Γ is a cocycle, we know that the principal face integrals contribute zero to the right-hand quantity. While the vanishing along hidden faces and faces at infinity indeed happens for $n > 3$, there is one type of face for which this fails in the case of defect zero and $n = 3$. This is known as the *anomalous face* and is indexed by all vertices of a connected component of a diagram colliding at the same point in $\mathbb{R}^n$. To fix this, one introduces a correction term which we give for the convenience of the reader in Eq. (40). This correction was first given by Bott and Taubes [5] in the case of knots and was generalized to links in [31, Theorem 4.5].

Remarks 5.1. (i) The collision of all configuration points can only take place in the space

$$C[0, \ldots, 0, k_j, 0, \ldots, 0; L, \Gamma], \quad 1 \leq j \leq m,$$

because points on different strands of a link cannot come together. The diagram Γ which corresponds to this situation thus must have a connected component with all its segment vertices on a single segment (and does not contain chords — if it does, the integral along the anomalous face vanishes; see [31, Proposition 4.3]). Since the integral associated to such a Γ computes a form on the space of knots (i.e. only on the jth strand of the link), the issue with anomalous faces is thus purely a knotting phenomenon, rather than a linking one.

(ii) As a consequence of the previous remark, and as was mentioned in the proof of Proposition 4.31, anomalous faces are thus not an issue for homotopy links. Because of how the complex $\mathcal{HD}$ is defined, a homotopy link diagram concentrated on one segment must be the empty diagram. The pushforward $\pi_{\mathcal{H},\Gamma}$ along the anomalous face thus vanishes and this is why $I_{\mathcal{H}}$ does not require a correction factor in Theorem 5.8.

To give the complete picture, we remind the reader of what the correction for the case of links is: Let sym_{S^2} now be a *rotation-invariant* smooth unit volume form on S^2. Also recall the definition of a connected component of a diagram (Definition 3.4), and let $\mathcal{LD}^0_{\mathrm{conn}}$ be the subcomplex of $\mathcal{LD}$ consisting of connected diagrams of defect zero (or degree zero, since $n = 3$). Consider the map

$$\tilde{I}_{\mathcal{L}} : \mathcal{LD}^0_{\mathrm{conn}} \to \Omega^0(\mathcal{L}^3_m)$$

defined as follows:

- If Γ
  - has segment vertices on only one segment, or;
  - has segment vertices on more than one segment but also contains a chord, then

$$(\tilde{I}_{\mathcal{L}})_\Gamma(L) = (I_{\mathcal{L}})_\Gamma(L);$$

- If Γ has segment vertices on only one segment, labeled s, and contains no chords, then

$$(\tilde{I}_{\mathcal{L}})_\Gamma(L) = (I_{\mathcal{L}})_\Gamma(L) - \mu_\Gamma \int_{C[2,L_s]} \left(\frac{x_1 - x_2}{|x_1 - x_2|}\right)^* \mathrm{sym}_{S^2}. \tag{40}$$

Here, L_s is the sth strand of the link L and μ_Γ is a real number which depends only on Γ and not on the link (this number is usually difficult to determine).

To extend $\tilde{I}_{\mathcal{L}}$ to a map

$$\tilde{I}_{\mathcal{L}} : \mathcal{LD}^0 \to \Omega^0(\mathcal{L}^3_m) \tag{41}$$

simply requires a little combinatorial organization. The reason is that if Γ has, say, two connected components Γ_1 and Γ_2, and configuration points corresponding to Γ_1 come together, the integral for this face is a product of two integrals,

$$(I_{\mathcal{L}})_{\Gamma_2} \cdot \partial_{\mathrm{anom}}(I_{\mathcal{L}})_{\Gamma_1},$$

where the second factor is the restriction of $(I_{\mathcal{L}})_{\Gamma_1}$ to the anomalous face. The correction for this term is thus

$$(I_{\mathcal{L}})_{\Gamma_2} \cdot \mu_{\Gamma_1} \int_{C[2,L_s]} \left(\frac{x_1 - x_2}{|x_1 - x_2|}\right)^* \mathrm{sym}_{S^2}.$$

However, one also has a situation when the roles of Γ_1 and Γ_2 are reversed, and further, each correction has its own anomalous face because of the first integral in the product. Thus, one has to account for correction terms of correction terms.

The pattern is clear if Γ has more than two connected components. Rather than writing this out, we refer the reader to the succinct formula for this iterated correction [26, Proposition 1.2] (this also appears in [1], but for framed knots). Even though this is a formula for knots and not links, understanding it for knots is sufficient by part (i) of Remark 5.1.

Again, by Stokes' Theorem, for any $\Gamma \in \mathcal{LD}^d_k$, $d(\tilde{I}_{\mathcal{L}}(\Gamma))$ can be written as an integral along the boundary of the fiber of $\oplus_l C[\vec{d}_l + s_l;\, \mathcal{L}^n_m, c_l(\Gamma)]$. This integral along the boundary can be broken up into contributions from principal faces, hidden faces, and faces at infinity. Using the proof of Proposition 4.31, which provides the erratum to [31] regarding faces at infinity, we have the following.

Theorem 5.2 ([31, Theorem 4.5]). *The contribution to $d(\tilde{I}_{\mathcal{L}}(\Gamma))$ from any hidden face (including any anomalous face) or any face at infinity is zero.*

Remark 5.3. We again wish to emphasize that, for this theorem to be true, it is important that we start with a rotation-invariant form sym_{S^2} on S^2. For details on why this is necessary, see [5, Lemma 5.7] (which uses Lemma 5.3, which in turn uses rotation invariance).

5.2. *Finite type invariants and chord diagrams*

We now briefly review the theory of finite type link invariants and recall how it is connected to the combinatorics of chord diagrams. Literature on this subject is abundant, but a good start for the case of knots is [3]. For a slightly more detailed overview than we give here for the case of links, see [31, Sec. 4.3].

Suppose we are given a link or a homotopy link invariant V, so that V is an element of $\mathrm{H}^0(\mathcal{L}^3_m)$ or $\mathrm{H}^0(\mathcal{H}^3_m)$. This invariant can be extended to *singular* links, by which we mean links with finitely many double-point self intersections where the two derivatives are independent. The singularities for ordinary links can come from a single strand crossing itself or two different strands intersecting. For homotopy links, we only consider those singularities arising from two different strands (if there is a singularity on a single strand, we ignore it). The extension of V is defined via the skein relation given in Fig. 22. The orientation on the link, which for us is given by the natural orientation of each of the m copies of $\mathbb{R}$, needs to be emphasized so that the two resolutions can be distinguished from each other (otherwise the two pictures on the right-hand side of the equation in Fig. 22 can be rotated into one another).

A k-singular link (a link with k singularities) thus produces 2^k links on which V can be evaluated. We will call these the *resolutions* of a singular link. Because of the signs, the order in which singularities are resolved does not matter.

Definition 5.4. The invariant V is *finite type k* (or *Vassiliev of type k*) if it vanishes on links with $k+1$ singularities.

Let

$$\mathcal{LV}_k = \text{real vector space generated by finite type } k \text{ link invariants;}$$

$$\mathcal{HV}_k = \text{real vector space generated by finite type } k \text{ homotopy link invariants.}$$

Note that $\mathcal{LV}_{k-1} \subset \mathcal{LV}_k$ and $\mathcal{HV}_{k-1} \subset \mathcal{HV}_k$ so that it makes sense to form quotients $\mathcal{LV}_k/\mathcal{LV}_{k-1}$ and $\mathcal{HV}_k/\mathcal{HV}_{k-1}$.

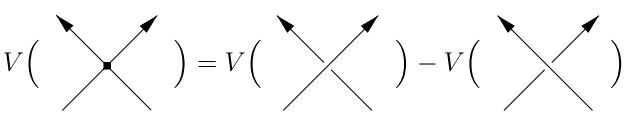

Fig. 22. Skein relation.

Next we want to describe a map f which to a finite type invariant associates a weight system (see Definition 3.35). Recall that we think of a weight system (as is usual) as a functional on diagrams satisfying the usual STU, IHX, and $1T$ relations. The construction is standard in finite type knot theory and this map is in fact the first connection between finite type invariants and the combinatorics of chord diagrams described in Sec. 3.4 (a detailed account of this in the case of knots is given in [3]). Here, we recall and adapt it to the setting of homotopy links. The inverse of f is given precisely by configuration space integrals and this is how one obtains isomorphisms in Theorems 5.6 and 5.8. The former was already proven in [31] so we will only provide a proof for the latter here.

Remark 5.5. Another way to construct an inverse to f is the famous *Kontsevich Integral* [14]. In fact, this integral provided the first proof of the isomorphism from Theorem 5.6 in the case of knots, i.e. when $m = 1$. This is known as the Fundamental Theorem of Finite Type Invariants.

To define f, first recall Theorem 3.38 and the terminology introduced after its statement. Let Γ be a chord diagram in $(\mathcal{HC}_k^0)^*$ and let H_Γ be any singular homotopy link with singularities as prescribed by Γ. By this we mean that H_Γ is any smooth map of m copies of $\mathbb{R}$ in $\mathbb{R}^3$ with, as usual, disjoint images and which is fixed outside a compact set, but which also has k "nice" self-intersections (locally embedded, derivatives independent at intersection point) given by $H_\Gamma(x_i) = H_\Gamma(y_j), x_i, y_j \in \mathbb{R}$, if there is a chord between vertices x_i and y_j in Γ. The points $H_\Gamma(x_i)$ and $H_\Gamma(y_j)$ are required to be on the strands corresponding to the segments that vertices x_i and y_j are on, and if x_i (y_j) comes before some other segment vertex $x_{i'}$ ($y_{j'}$) in the ordering of the vertices of Γ (we picture x_i as lying to the left of $x_{i'}$ in this case), then $x_i < x_{i'}$ ($y_j < y_{j'}$) as points in $\mathbb{R}$ (by abuse of notation, we label the segment vertices the same way as coordinates in $\mathbb{R}$). An example is given in Fig. 23.

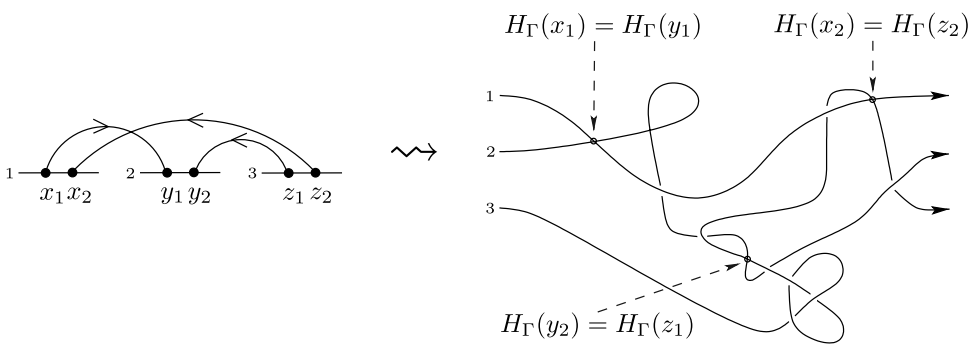

Fig. 23. An example of a homotopy link H_Γ associated to a chord diagram $\Gamma \in (\mathcal{HC}_3^0)^*$. The only requirement is that the relative positions of the singularities respect the relative positions of the chords. Note that strand 2 intersects itself but we ignore such singularities.

Now consider the value of a type k invariant $V \in \mathcal{HV}_k$ on (the sum of the resolutions of) H_Γ. This value remains unchanged if a crossing between two strands of H_Γ is switched because, by the skein relation,

$$V(H_\Gamma) - V(H_\Gamma \text{ with a crossing changed}) = V(\text{some } (k+1)\text{-singular link}) = 0.$$

This means that V does not depend on a particular link but only on the placement of singularities. It thus makes sense to define a map

$$\begin{aligned} f : \mathcal{HV}_k &\to \mathcal{HCW}_k, \\ V &\mapsto \begin{pmatrix} W : (\mathcal{HC}_k^0)^*/(4\mathrm{T}, 1\mathrm{T}) \to \mathbb{R} \\ \Gamma \mapsto V(H_\Gamma) \end{pmatrix}. \end{aligned}$$

It follows immediately from the definitions that the kernel of f consists precisely of type $k-1$ invariants, so that f becomes an injection

$$f : \mathcal{HV}_k/\mathcal{HV}_{k-1} \hookrightarrow \mathcal{HCW}_k. \tag{42}$$

We can then use the isomorphism

$$\mathcal{HW}_k \cong \mathcal{HCW}_k$$

from (17) to extend f to a functional on trivalent diagrams which satisfies the usual STU, IHX, and $1T$ relations. (Recall that this isomorphism is induced by sending a chord diagram to itself and trivalent diagram to a sum of chord diagrams obtained from it by resolving all the free vertices via the STU relation.) We obtain then an extension of f to an injection

$$\begin{aligned} f : \mathcal{HV}_k/\mathcal{HV}_{k-1} &\hookrightarrow \mathcal{HW}_k, \\ V &\mapsto \begin{pmatrix} W : \mathcal{HD}_k^0/(STU, IHX, 1T) \to \mathbb{R} \\ \Gamma \mapsto \begin{cases} V(H_\Gamma), & \Gamma \text{ chord diagram;} \\ \sum_i V(H_{\Gamma_i}), & \Gamma \text{ trivalent diagram} \end{cases} \end{pmatrix}, \end{aligned} \tag{43}$$

where the Γ_i are the chord diagram resolutions of a trivalent diagram Γ.

5.3. *Integrals and finite type invariants of homotopy string links*

We are now almost ready to state and prove the main result of this section, Theorem 5.8. This theorem states that configuration space integrals give an isomorphism between weight systems and finite type invariants of homotopy links. We will show this by exhibiting the map f above as the inverse to integration. The integration of weight systems is essentially the same as the integration map $I_{\mathcal{H}}$ from the graph complex. Before stating the theorem, we explain how this works.

We can consider $\mathcal{HW}_k \cong ((\mathcal{HD}_k^0)^*/(STU, IHX, H1T))^*$ as a space of functionals on $(\mathcal{LD}_k^0)^*$ satisfying certain relations (since $(\mathcal{HD}_k^0)^*/(STU, IHX, H1T)$ is

a quotient of $(\mathcal{LD}_k^0)^*$). Choose a basis $\mathcal{B}_k$ of diagrams for $(\mathcal{LD}_k^0)^*$. Certainly $\mathcal{B}_k$ is finite (and it is canonical up to signs of the elements). Since $\mathcal{LW}_k \cong \mathrm{H}^0(\mathcal{LD}_k^*)$ canonically, a weight system W corresponds canonically to some linear combination of diagrams $\sum_{\Gamma\in\mathcal{B}_k} a_\Gamma \Gamma$.

Thus, we have a composition

$$\mathcal{LW}_k \xleftrightarrow{\cong} \mathrm{H}^0(\mathcal{LD}_k^*) \xrightarrow{I_{\mathcal{L}}^0} \Omega^0(\mathcal{L}_m^3)$$

given by

$$W \longleftrightarrow \sum_{\Gamma\in\mathcal{B}_k} a_\Gamma \Gamma \longmapsto \sum_{\Gamma\in\mathcal{B}_k} a_\Gamma (I_{\mathcal{L}})_\Gamma.$$

Since for all $\Gamma \in \mathcal{B}_k, W(\Gamma) = \langle a_\Gamma\Gamma, \Gamma\rangle = a_\Gamma|\mathrm{Aut}(\Gamma)|$, we have $a_\Gamma = W(\Gamma)/|\mathrm{Aut}(\Gamma)|$.

So we can rewrite this composition as

$$W \mapsto \sum_{\Gamma\in\mathcal{B}_k} \frac{W(\Gamma)}{|\mathrm{Aut}(\Gamma)|}(I_{\mathcal{L}})_\Gamma. \tag{44}$$

The latter expression is similar to one of the two formulae for producing a knot invariant from a weight system via configuration space integrals originally written down in [29]. The only difference is that our formula above contains no anomaly term because W is an element of $\mathcal{HW}_k$, rather than an arbitrary element of $\mathcal{LW}_k$. For ordinary link invariants (including knot invariants), all of the above paragraph applies, except that we would have to use the correction for the anomaly term $\widetilde{I}_{\mathcal{L}}$ instead of the map $I_{\mathcal{L}}$. This is what we do in the statement of Theorem 5.6.

The second formula in [29] equivalent to (44) is a sum over *labeled* diagrams in which the $|\mathrm{Aut}(\Gamma)|$ factors do not appear. This latter formula (which also appears in [30]) is not immediately compatible with integration from the graph complex because in the graph complex, diagrams with different labels are equal up to sign. Nonetheless, the above description as a sum over unlabeled diagrams should clarify the relationship between integration of *weight systems* (i.e. functionals) for finite type invariants (as in [29, 30]) and integration from the graph complex of *diagrams* (as in [7]).

The following statement for links already appeared as [31, Theorems 4.7 and 4.11], though the correct proof of those theorems requires the erratum we provided in proving Proposition 4.31.

Theorem 5.6. *For $k \geq 0$ and $m \geq 1$, the map*

$$I_{\mathcal{L}}^0 : \mathcal{LW}_k \to \mathcal{LV}_k$$

given by

$$W \mapsto \left(L \mapsto \sum_{\Gamma\in\mathcal{B}_k} \frac{W(\Gamma)}{|\mathrm{Aut}(\Gamma)|}(\tilde{I}_{\mathcal{L}})_\Gamma(L) \right)$$

gives a section to the natural projection $\mathcal{LV}_k \to \mathcal{LV}_k/\mathcal{LV}_{k-1} \cong \mathcal{LW}_k$.

Remark 5.7. Note that the map $I^0_{\mathcal{L}}$ exists even for $n > 3$. However, one then obtains cohomology classes of $\mathcal{L}^n_m$ in degree $(n-3)k$ rather than in degree zero. The same is true for the map $I^0_{\mathcal{H}}$ in Theorem 5.8.

We now prove the same statement for homotopy links.

Theorem 5.8. *For $k \geq 0$ and $m \geq 1$, the map*

$$I^0_{\mathcal{H}} : \mathcal{HW}_k \to \mathcal{HV}_k$$

given by

$$W \mapsto \left(H \mapsto \sum_{\Gamma \in \mathcal{B}_k} \frac{W(\Gamma)}{|\mathrm{Aut}(\Gamma)|} (I_{\mathcal{H}})_\Gamma(H) \right)$$

gives a section to the natural projection $\mathcal{HV}_k \to \mathcal{HV}_k/\mathcal{HV}_{k-1} \cong \mathcal{HW}_k$.

Remark 5.9. The sums in the two theorems above are both taken over the basis $\mathcal{B}_k$ for $(\mathcal{LD}^0_k)^*$, though equivalently, one could remove from $\mathcal{B}_k$ all the Γ such that the $1T$ relation (or its homotopy link analogue) forces $W(\Gamma) = 0$; this subset of $\mathcal{B}_k$ will be smaller for $\mathcal{HW}$ than for $\mathcal{LW}$.

We first prove part of this theorem in the following.

Proposition 5.10. *The image of $I^0_{\mathcal{H}}$ is a subset of $\mathcal{HV}_k$.*

Proof. We have a commutative diagram as below. The inclusion $\mathcal{HW}_k \to \mathcal{LW}_k$ just comes from the fact that an element satisfying the relations defining $\mathcal{HW}_k$ must satisfy the weaker relations defining $\mathcal{LW}_k$. The rest of the diagram is the square (37), where we use Theorem 5.6 to deduce that the middle map in the bottom row is injective:

$$\begin{tikzcd}
\mathcal{HW}_k \arrow[r, leftrightarrow, "\cong"] \arrow[d, hook] & \mathrm{H}^0(\mathcal{HD}^*_k) \arrow[rr, "I^0_{\mathcal{H}}"] \arrow[d, hook] & & \mathrm{H}^0(\mathcal{H}^3_m) \arrow[d, hook] \\
\mathcal{LW}_k \arrow[r, leftrightarrow, "\cong"] & \mathrm{H}^0(\mathcal{LD}^*_k) \arrow[r, hook, "I^0_{\mathcal{L}}"] & \mathcal{LV}_k \arrow[r, hook] & \mathrm{H}^0(\mathcal{L}^3_m)
\end{tikzcd} \tag{45}$$

As shown in the diagram, we already know that elements in the image of $I^0_{\mathcal{H}}$ are invariants of homotopy links, since $I^0_{\mathcal{H}}$ is a chain map. Furthermore, invariants in the image of $I^0_{\mathcal{H}}$ are finite type since their image under the rightmost vertical map is in $\mathcal{LV}_k$. This proves the desired statement. □

Remark 5.11 (Explicit proof of link-homotopy invariance). From Theorem 4.33, we already knew that the link invariants in the image of $I^0_{\mathcal{H}}$ are invariant under link homotopy. We now present a hands-on, concrete proof of this fact; since this is an argument reminiscent of the original proofs that Bott–Taubes integrals produce finite type invariants, we think this might be beneficial for the reader.

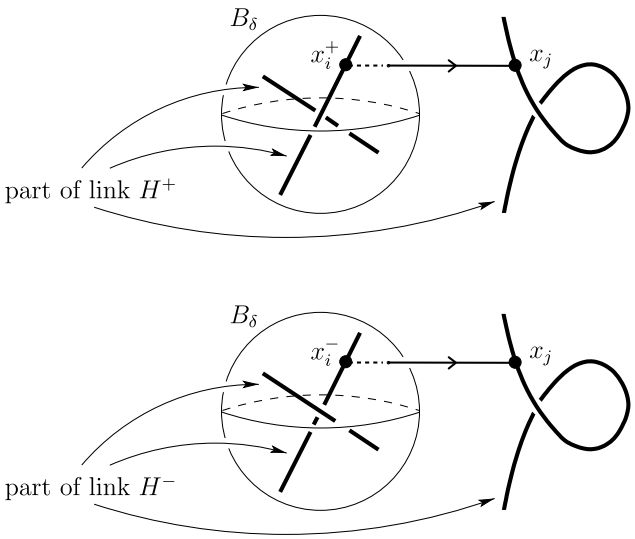

Fig. 24. Homotopy links H^+ and H^- are the same outside the ball B_δ where they differ as pictured. The two arcs in B_δ come from the same strand.

By the discussion at the end of Sec. 2, it suffices to show that this integral takes the same value on a link before and after a crossing change. Thus, it suffices to show that given a diagram $\Gamma \in \mathcal{HD}_k$ and links H^+, H^- which differ only inside a ball B_δ or radius δ as pictured in Fig. 24, we have

$$(I_{\mathcal{H}})_\Gamma(H^+) = (I_{\mathcal{H}})_\Gamma(H^-).$$

In other words,

$$\int_{\oplus_l C[\vec{d}_l + s_l; H^+, c_l(\Gamma)]} \prod_{\text{edges } (a,b) \text{ of } \Gamma} \left(\frac{x_a - x_b}{|x_a - x_b|}\right)^* \mathrm{sym}_{S^2}$$
$$- \int_{\oplus_l C[\vec{d}_l + s_l; H^-, c_l(\Gamma)]} \prod_{\text{edges } (a,b) \text{ of } \Gamma} \left(\frac{x_a - x_b}{|x_a - x_b|}\right)^* \mathrm{sym}_{S^2} = 0. \qquad (46)$$

As usual, the configuration points x_a and x_b here correspond to diagram vertices a and b.

The domain of integration over which the two integrals differ has measure a constant times δ, and the integrals over these regions are bounded since $|x_a - x_b| > \epsilon > 0$ for some ϵ independent of δ for all a and b because such x_a and x_b will never lie on the same strand. It follows that the difference of the integrals can be made arbitrarily small.

Proof of Theorem 5.8. To show that $I^0_{\mathcal{H}}$ is an isomorphism, we argue that its inverse is the map

$$f : \mathcal{HV}_k / \mathcal{HV}_{k-1} \hookrightarrow \mathcal{HW}_k$$

from (43). We claim it suffices to prove that the composition

$$\mathcal{HW}_k \xrightarrow{I^0_{\mathcal{H}}} \mathcal{HV}_k/\mathcal{HV}_{k-1} \xhookrightarrow{f} \mathcal{HW}_k \tag{47}$$

is the identity. In fact, $f \circ I^0_{\mathcal{H}} = \mathrm{id}$ will imply $f \circ (I^0_{\mathcal{H}} \circ f) = f$, and since f is injective, it will follow that $I^0_{\mathcal{H}} \circ f = \mathrm{id}$. Furthermore, by the isomorphism $\mathcal{HCW}_k \cong \mathcal{HW}_k$ we may think of the composition above as

$$\mathcal{HCW}_k \xrightarrow{I^0_{\mathcal{H}}} \mathcal{HV}_k/\mathcal{HV}_{k-1} \xhookrightarrow{f} \mathcal{HCW}_k. \tag{48}$$

To describe this composition, we choose a singular (homotopy) link H_Γ for each chord diagram Γ with k chords. The (labeled) singularities in H_Γ will be prescribed by Γ, much like in the discussion preceding Fig. 23. In this setting of invariants of link homotopy, we can actually construct the H_Γ's quite explicitly.

We start the construction with a trivial string link with its segments all horizontal, numbered in decreasing order of y-coordinate.[c] More precisely, we start with m disjoint copies of $\mathbb{R}$, where inside some interval $[-t_0, t_0] \subset \mathbb{R}$ the ith strand is given by $t \mapsto (t, -i, 0)$, and outside a larger interval $[-t_1, t_1]$, the ith strand is given by $t \mapsto (t, |t|(\frac{m+1}{2} - i), 0)$.

In what follows, "above" (respectively, "below") will mean above (respectively, below) the xy-plane in $\mathbb{R}^3$. We manipulate the strands (within $\sqcup_m[-t_0, t_0] \subset \sqcup_m\mathbb{R}$) to make the singular link H_Γ, as follows. If there is a chord between the ith and jth strands and $i < j$, then move strand i so that it passes below strands $i+1, \ldots, j-1$, intersect strand j in a single point, passes beneath strands $j, j-1, \ldots, i+1$, and then resumes its course along $\{(x, -i, 0)\}$.

Figure 25 shows a picture of such an H_Γ. (We only show the image of the smaller intervals $[-t_0, t_0] \subset \mathbb{R}$. Recall that different directions toward infinity, outside of $[-t_1, t_1]$, were required for certain evaluation maps, and hence the configuration space integrals, to be well-defined.)

By the Vassiliev skein relation (Fig. 22), the value of a type k invariant on H_Γ is its value on a signed sum of the 2^k resolutions of H_Γ. It will be useful to define specific resolutions of H_Γ, one resolution H^S_Γ for each $S \subseteq \{1, 2, \ldots, k\}$. We can take H_Γ (and hence each H^S_Γ) to lie in the xy-plane, except for crossings which take place inside small balls. Each H^S_Γ will agree with H_Γ outside of k small balls around the k double points of H_Γ.

Define the link H^S_Γ as follows: Consider a chord in Γ corresponding to an $\ell \in \{1, \ldots, k\}$. This also corresponds to a double point in H_Γ. Let $i < j$ be the endpoints of the chord ℓ. If $\ell \in S$, then in H^S_Γ, resolve the double point ℓ by perturbing strand

[c]These are not quite the "horizontal" or "tangle" chord diagrams considered by some authors. The reason is that there could be two chords between two strands that cross and there is no way to draw all chords horizontally in such a situation. However, weight systems that are associated with Milnor invariants vanish on chord diagrams with more that one chord connecting two segments [19], so in that case one can reduce to the case of genuine tangles. More will be said about this in future work.

i slightly so that it goes over strand j. If $\ell \notin S$, then perturb strand i slightly so that it goes under strand j. Perhaps a better way of thinking about this is as follows: each H_Γ^S is the resolution of H_Γ where each double point in S has been resolved "positively" (as in the first picture on the right-hand side of the equation in Fig. 22), while the remaining singularities have been resolved "negatively" (as in the second picture on the right-hand side of the equation in Fig. 22). An example is shown in Fig. 25.

Returning to the composition (48), it is given by

$$(\Gamma \mapsto W(\Gamma)) \mapsto \left(\Gamma \mapsto \sum_{\Gamma' \in \mathcal{B}_k} W(\Gamma') \sum_{S \subseteq \{1,2,\ldots,k\}} (-1)^{k-|S|} (I_{\mathcal{H}})_{\Gamma'} (H_\Gamma^S)\right),$$

whereas before, $\mathcal{B}_k$ is a basis of trivalent diagrams (canonical up to the sign of each diagram). Here, $W \in \mathcal{HCW}_k$ is determined on arbitrary trivalent diagrams Γ' by the STU relation.

This composition will be the identity if we can show that

$$I_{\Gamma'} := \sum_{S \subset \{1,2,\ldots,k\}} (-1)^{k-|S|} (I_{\mathcal{H}})_{\Gamma'} (H_\Gamma^S) = \begin{cases} 1, & \Gamma' = \Gamma, \\ 0, & \Gamma' \neq \Gamma. \end{cases} \tag{49}$$

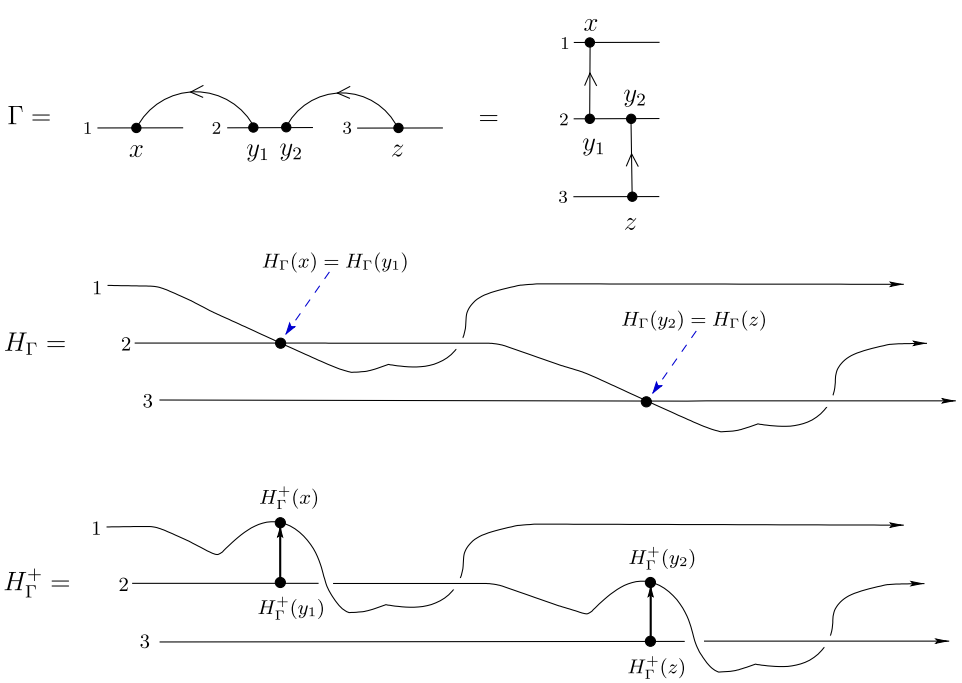

Fig. 25. An example of a diagram Γ, its horizontal version, a singular homotopy link H_Γ, and its resolution $H_\Gamma^+ := H_\Gamma^{\{1,\ldots,k\}}$ for which $(I_{\mathcal{H}}^0)_\Gamma$ is nonzero.

First, let $\mathcal{C}$ be the subspace of configuration space where there are exactly two points in the ball around each resolved double point. We first consider the case where Γ' is a chord diagram. We consider the contributions to $I_{\Gamma'}$ from integrating over $\mathcal{C}$, then show that the integral over the complement of $\mathcal{C}$ is zero.

If Γ' is a chord diagram other than Γ, then $\mathcal{C}$ is empty.

If $\Gamma = \Gamma'$, the signed sum of integrals I'_Γ over the subspace $\mathcal{C}$ can be rewritten as a sum of integrals with all $+1$ signs by reversing the orientation on every "understrand" in H^S_Γ. The resulting sum is the integral over a configuration space of points on circles enclosing (straight) arcs. By choosing smaller perturbations of the strands in all the resolutions, this integral can be made arbitrarily close to a product of linking numbers, one for each circle-arc pair. See Fig. 26. But the linking number of each such pair is $+1$ (assuming appropriate choices of orientation on the configuration space and the k spheres from which the integrand is pulled back). Thus, the integral I_Γ over $\mathcal{C}$ can be made arbitrarily close to 1.

For any chord diagram Γ', the integral $I_{\Gamma'}$ over the complement of $\mathcal{C}$ vanishes. Indeed, for any configuration in the complement of $\mathcal{C}$, there will be some pair of points joined by a chord $\ell \in \{1, \ldots, k\}$ where at least one of the points is outside the ball around the resolved double point ℓ. Partitioning the 2^k terms into two parts according to the sign of this ℓth resolution, one can show that the two parts nearly cancel; that is, by making the balls smaller, these contributions can be made arbitrarily close to 0. The details are similar to the proof of [30, Lemma 5.4], though arguably simpler because our H^S_Γ's are "almost horizontal".

Finally, if Γ' is not a chord diagram, then the integral $I_{\Gamma'}$ over $\mathcal{C}$ is also arbitrarily close to 0 because the contributions over the sum of the 2^k terms can be similarly cancelled in pairs. The details are exactly as given in [30], at the end of the proof of Lemma 5.4 of that paper.

Now the integral $I_{\Gamma'}$ is an isotopy invariant. Thus, in the arguments above, we may replace "arbitrarily close to" by "equal to". So the only nonzero contribution to $I_{\Gamma'}$ is when $\Gamma = \Gamma'$ in which case $I_{\Gamma'} = 1$. This proves (49), which completes the proof of the theorem. □

Remark 5.12. Even though in the proof of Proposition 5.10 we appealed to Theorem 5.6 and the fact that $I^0_{\mathcal{L}}$ is a universal finite type invariant of ordinary string links, it is easy to prove Proposition 5.10 in a way that is independent of Theorem 5.6. In addition, the proof of Theorem 5.8 essentially works the same way for

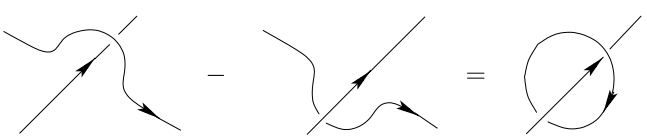

Fig. 26. A schematic if how, for $\Gamma = \Gamma'$, the integral $I_{\Gamma'}$ over the subspace $\mathcal{C}$ is a product of linking numbers of circles with arcs.

string links as it does for homotopy string links. In light of the fact that the proof of Theorem 5.6 is only outlined in [31] (and requires the erratum from our Proposition 4.31 to see that the integration map gives link invariants), one can thus regard the complete picture given here for finite type invariants of homotopy string links as also giving a fairly complete picture of finite type invariants for ordinary string links.

5.4. *Milnor invariants of homotopy string links*

With Theorem 5.8 in hand, we can now quickly deduce the corollary about Milnor invariants of string links as promised in Sec. 1.

For m-component string links, each non-repeating index Milnor invariant $\mu_{i_1 i_2 \cdots i_{k+1}}, 1 \leq i_j \leq m$, is well-defined (for closed links, there is an indeterminacy, modulo which one gets the $\overline{\mu}$ invariants), and it is a finite type k invariant [4, 16]. Furthermore, this is a link-homotopy invariant [21]. Thus, $\mu_{i_1 i_2 \cdots i_{k+1}}$ can be thought of as a finite type invariant of $\mathcal{H}^3_m$ (here, we again use the discussion following Corollary 2.4).

We have then the following consequence of Theorem 5.8.

Theorem 5.13. *Each Milnor invariant $\mu_{i_1 i_2 \cdots i_{k+1}}$ of string links of m components is given, up to a type $(k-1)$ invariant, by*

$$\mu_{i_1 i_2 \cdots i_{k+1}}(H) = (I^0_{\mathcal{H}}(W))(H) = \sum_{\Gamma \in \mathcal{B}_k} \frac{W(\Gamma)}{|\mathrm{Aut}(\Gamma)|} (I_{\mathcal{H}})_\Gamma(H) \tag{50}$$

for some weight system $W \in \mathcal{HW}_k$, where $\mathcal{B}_k$ is a basis of diagrams for $(\mathcal{HD}^0_k)^*$.

We can refine this statement. If $k+1 < m$, then some index j between 1 and m does not appear in the subscript of $\mu_{i_1 i_2 \cdots i_{k+1}}$, and we then have a Milnor invariant of $(m-1)$-component links, namely an invariant of the link obtained by deleting the jth strand. By relabeling, we can assume that the deleted strand is in fact the mth one. To understand Milnor invariants, it suffices to study those invariants of m-component links that are not induced by the projection

$$\mathcal{H}^3_m \to \mathcal{H}^3_{m-1}$$

given by deleting the mth strand of a link. This means that, in the sum from (50), we only take those diagrams Γ with segment vertices appearing on *all* segments. If the sum is taken over only those diagrams that do not have any segment vertices on, say, the mth segment, then one obtains an invariant of $(m-1)$-component links. This is easy to see as such diagrams account for all the necessary cancellations of integration along faces and thus produce a closed form. We will call diagrams with segment vertices on all segments *maximal* and will denote them by $\Gamma_{\max}$.

It follows that, since $\mu_{i_1 i_2 \cdots i_m}$ is a type $m-1$ invariant, each $\Gamma_{\max}$ must have $2(m-1)$ vertices, at least m of which are segment vertices, lying on m segments. These can also be characterized as forests with at least m but no more than

$2(m-1)$ leaves with m distinct labels (each label is associated with a unique segment/strand). Recall that by a forest we mean a disjoint union of trees, and by a tree we mean the collection of vertices and edges, but not segments, of a diagram, where the leaves are the segment vertices.

We thus get the following corollary.

Corollary 5.14. *Each Milnor invariant $\mu_{i_1 i_2 \cdots i_m}$ of string links of m components is given, up to a type $(m-2)$ invariant, by*

$$\mu_{i_1 i_2 \cdots i_m}(H) = (I^0_{\mathcal{H}}(W))(H) = \sum_{\Gamma_{\max} \in \mathcal{B}_{m-1}} \frac{W(\Gamma)}{|\mathrm{Aut}(\Gamma)|} (I_{\mathcal{H}})_{\Gamma_{\max}}(H) \tag{51}$$

for some weight system $W \in \mathcal{HW}_{m-1}$, where $\mathcal{B}_{m-1}$ is a basis of diagrams for $(\mathcal{HD}^0_{m-1})^*$.

Remark 5.15. Suppose that in addition we required that $\Gamma_{\max} \in (\mathcal{HD}_{m-1})^*$ be connected. It is immediate that such a trivalent diagram must have precisely m segment vertices (one on each of the m segments) and $m-2$ free vertices. Since diagrams in $(\mathcal{HD}_m)^*$ have no loops of edges, it follows that a connected $\Gamma_{\max}$ is precisely a tree with m leaves.

The next step is to understand precisely which weight systems appear in Corollary 5.14. In particular, one could to this end utilize the combinatorial properties of such "Milnor weight systems" established in [19]. The connection to [12] should also be explored; one of the results of that paper is that Milnor invariants of string links correspond to the tree part of the Kontsevich integral, and it is this integral that gives an alternative way of showing that weight systems correspond to finite type invariants. (In fact, the Kontsevich integral provided the first proof of this theorem.) In addition, as mentioned in Sec. 1, a further study of configuration space integrals and Milnor invariants in the context of manifold calculus of functors could also be beneficial.

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