Configuration space integrals and the cohomology of knot and link spaces

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Configuration spaces, braids and applications
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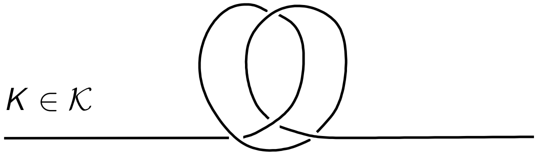
Outline of the talk

1. Motivation: The linking number
2. Configuration space integrals for knots and finite type invariants
3. Generalizations and applications
   (a) Generalization to cohomology of knot spaces in dimension > 3
   (b) Generalization to spaces of links and homotopy links (and braids)
   (c) Configuration space integrals and calculus of functors
   (d) Configuration space integrals and rational homotopy theory

Space of long knots in $\mathbb{R}^3$

$$\mathcal{K} = \{\text{embeddings } \mathbb{R} \hookrightarrow \mathbb{R}^3 \text{ fixed outside a compact set}\}$$
$$= \textit{space of long knots}$$



We are interested in computing

$$\mathrm{H}_0(\mathcal{K}) = \{\text{knot types}\} = \{\text{isotopy classes of knots}\}$$
$$\mathrm{H}^0(\mathcal{K}) = \{\text{knot invariants } f\colon \mathrm{H}_0(\mathcal{K}) \to \mathbb{R}\}$$

This is classical knot theory. Related space is

$$\mathcal{L}_2 = \{\text{embeddings } \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}^3, \text{ fixed outside a compact set}\}$$
$$= \textit{space of long (string) links of two components}$$

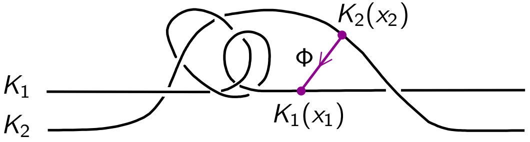
1. Linking number

Let $\mathrm{Conf}(k, \mathbb{R}^n) = \{(a_1, a_2, ..., a_k) \in \mathbb{R}^{kn} : a_i \neq a_j \text{ for } i \neq j\}$
$= \textit{configuration space of } k \textit{ points in } \mathbb{R}^n$

Consider the maps Φ and π:

$$\Phi\colon \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 \xrightarrow{\textit{evaluation}} \mathrm{Conf}(2, \mathbb{R}^3) \xrightarrow{\textit{direction}} S^2$$

$$(x_1, x_2, L = (K_1, K_2)) \longmapsto (K_1(x_1), K_2(x_2)) \longmapsto \frac{K_2(x_2) - K_1(x_1)}{|K_2(x_2) - K_1(x_1)|}$$



$$\pi\colon \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 \xrightarrow{\textit{projection}} \mathcal{L}_2 \quad \text{(trivial bundle)}$$

1. Linking number

So have a diagram

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 & \xrightarrow{\Phi} & S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{L}_2 & & \end{array}$$

which, on the complex of deRham cochains (differential forms), gives a diagram

$$\begin{array}{ccc} \Omega^*(\mathbb{R} \times \mathbb{R} \times \mathcal{L}_2) & \xleftarrow{\Phi^*} & \Omega^*(S^2) \\ \downarrow{\scriptstyle \pi_*} & & \\ \Omega^{*-2}(\mathcal{L}_2) & & \end{array}$$

Here Φ^* is the usual pullback and π_* is *integration along the fiber*, or *pushforward* – a way to create forms on the base space of a bundle from forms on the total space, shifted by the dimension of the fiber.

1. Linking number

Let $sym_{S^2} \in \Omega^2(S^2)$ be the unit volume form on S^2. Then the linking number is

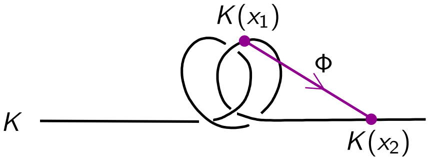
$$\boxed{Link(K_1, K_2) = \pi_*(\Phi^*(sym_{S^2})) = \int_{\mathbb{R}\times\mathbb{R}} \Phi^*(sym_{S^2}) \in \Omega^0(\mathcal{L}_2)}$$

This is indeed a closed form, i.e. an element of $\mathrm{H}^0(\mathcal{L}_2)$, and hence an invariant of two-component links (this goes back to Gauss).

Now try to do the same, but for a single knot rather than a link.

2. Configuration space integrals: try to mimic $lk(K_1, K_2)$

The picture is



And the corresponding diagram is

$$\begin{array}{ccc} \mathrm{Conf}(2,\mathbb{R}) \times \mathcal{K} & \xrightarrow{\Phi} & S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{K} & & \end{array}$$

The first issue is that an integral over $\mathrm{Conf}(2,\mathbb{R})$ may not converge since this space is open. So need to compactify.

2. Configuration space integrals: Fulton-MacPherson compactification

Definition

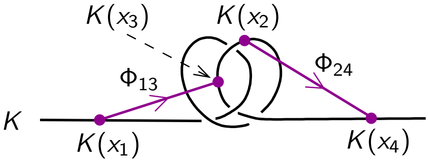
Let $\mathrm{Conf}[k, \mathbb{R}^n]$ be the Fulton-MacPherson compactification of $\mathrm{Conf}(k, \mathbb{R}^n)$.

Some properties:

- $\mathrm{Conf}[k, \mathbb{R}^n]$ is homotopy equivalent to $\mathrm{Conf}(k, \mathbb{R}^n)$;
- $\mathrm{Conf}[k, \mathbb{R}^n]$ is a manifold with corners;
- Boundary of $\mathrm{Conf}[k, \mathbb{R}^n]$ is characterized by points colliding with directions and relative rates of collisions kept track of;
- Codimension 1 boundary (important for Stokes' Theorem) given by points coming together at the same time (rather than in stages);
- $\mathrm{Conf}[k, \mathbb{R}]$ is the *associahedron*;
- Works for configurations in any manifold, not just $\mathbb{R}^n$.

2. Configuration space integrals: simplest case for knots

But, even after compactifying, we still do not get an invariant.
The next case is that of four points and two directions:



The maps are

$$\begin{array}{ccc} \mathrm{Conf}[4,\mathbb{R}] \times \mathcal{K}^3 & \xrightarrow{\Phi=\Phi_{13}\times\Phi_{24}} & S^2 \times S^2 \\ \Big\downarrow \pi & & \\ \mathcal{K}^3 & & \end{array}$$

Thus get a 0-form

$$I(\,$$  $$, K) = \pi_*(\Phi^*(sym^2_{S^2})) = \int_{\mathrm{Conf}[4,\mathbb{R}]} \Phi^*(sym^2_{S^2})$$

2. Configuration space integrals: simplest case for knots

So $I($$, K)$ is a 0-form, i.e. an element of $\Omega^0(\mathcal{K})$. But is is a closed form, that is an element of $\mathrm{H}^0(\mathcal{K})$ – an invariant?

So want $dI($$, K) = 0$. Stokes' Theorem says that

$dI($$, K)$

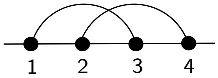
$$= \pi_*(d\Phi^*(sym_{S^2} \wedge sym_{S^2})) + (\partial\pi)_*(\Phi^*(sym^2_{S^2}))$$
$$= (\partial\pi)_*(\Phi^*(sym^2_{S^2})) \quad \text{(since } sym_{S^2} \text{ is closed)}$$

Here $(\partial\pi)_*(\Phi^*(sym^2_{S^2}))$ is the pushforward along codimension one faces of $\mathrm{Conf}[4, \mathbb{R}]$.

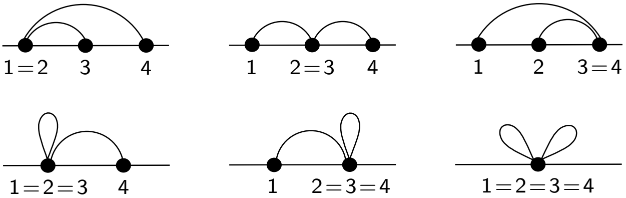
These faces can be represented by *diagrams* as follows.

2. Configuration space integrals: boundary diagrams

If there are four points moving on the knot, and two directions are kept track of as above, the diagram encoding this information is



Codimension one faces (collisions) are then encoded by diagrams



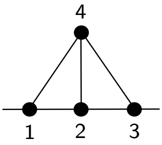
(Loop corresponds to the derivative map.)

It turns out that the integrals corresponding to the bottom three diagrams vanish, but not necessarily for the top three.

One way to resolve this: Look for another space to integrate over which has the same three faces and subtract the integrals.

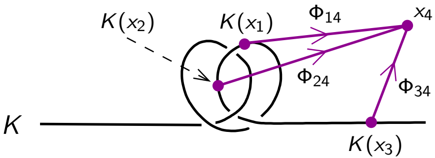
2. Configuration space integrals: the fix

The diagram that fits what we need is



since, when $4=1$, $4=2$, and $4=3$, we get the same three relevant pictures as before (up to relabeling).

This suggests that we want a space of four configuration points in $\mathbb{R}^3$, three of which lie on a knot. In other words, we want the following picture:

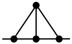


2. Configuration space integrals: the fix

To make this precise, define pullback space

$$\begin{array}{ccc} \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \longrightarrow & \mathrm{Conf}[4,\mathbb{R}^3] \\ \downarrow & & \downarrow \mathit{proj} \\ \mathrm{Conf}[3,\mathbb{R}]\times\mathcal{K} & \xrightarrow{\mathit{eval}} & \mathrm{Conf}[3,\mathbb{R}^3] \end{array}$$

The natural map $\pi\colon\ \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]\to\mathcal{K}$ with fiber $\mathrm{Conf}[3,1;K,\mathbb{R}^3]$ over K is a bundle (Bott-Taubes) so we can integrate along its fiber. Thus for each $K\in\mathcal{K}$, we get an integral

$I($  $, K) = \pi_*(\Phi^*(sym^3_{S^2})) = \int_{\mathrm{Conf}[3,1;K,\mathbb{R}^3]} \Phi^*(sym^3_{S^2})$

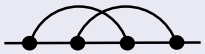
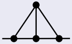
where $\Phi = \Phi_{14}\times\Phi_{24}\times\Phi_{34}$.

It turns out that the boundary contributions for this integrals are zero except for the three boundary pieces we care about.

2. Configuration space integrals: the fix

Theorem (Altschuler-Friedel, Bar-Natan)

The map

$K \longmapsto \Big(I($  $, K) - I($  $, K) \Big)$

is a knot invariant. Further, it is a finite type two invariant.

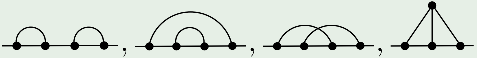
Finite type invariants are a class of invariants that has received much attention in recent years. It is conjectured to *separate* knots, i.e. to form a complete set of knot invariants. (It is known to separate braids and homotopy links – more about these later.)

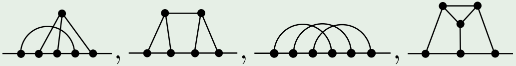
But there is no reason to stop at four configuration points...

2. Configuration space integrals: general case

$\mathcal{TD}_k = \{\mathbb{R}\text{-vect. sp. gen'd by } \textit{trivalent diagrams} \text{ with } 2k \text{ vertices}\}.$

Example

$\mathcal{TD}_2 = \{$  $\}$

$\mathcal{TD}_3 = \{$  , etc. $\}$

Given $D \in \mathcal{TD}_k$ with p vertices on the segment and q off the segment, consider the pullback

$$\begin{array}{ccc}
\mathrm{Conf}[p, q; \mathcal{K}, \mathbb{R}^3] & \longrightarrow & \mathrm{Conf}[p+q, \mathbb{R}^3] \\
\downarrow & & \downarrow \, \mathit{proj} \\
\mathrm{Conf}[p, \mathbb{R}] \times \mathcal{K} & \xrightarrow{\ \mathit{eval}\ } & \mathrm{Conf}[p, \mathbb{R}^3]
\end{array}$$

2. Configuration space integrals: general case

Then, for each $K \in \mathcal{K}$, have integral

$$\boxed{I(D,K) = \pi_*(\Phi^*(sym^e_{S^2})) = \int_{\mathrm{Conf}[p,q;K,\mathbb{R}^3]} \Phi^*(sym^e_{S^2})}$$

where e is the number of edges of E and Φ is the product of the direction maps between pairs of configuration points corresponding to the edges of D.
Let $\mathcal{W}_k = \mathcal{TD}^*_k/(STU, IHX)$.

Theorem (Bott-Taubes, D.Thurston)

Given $W \in \mathcal{W}_k$, the map

$$K \longmapsto \sum_{D \in \mathcal{TD}_k} W(D)\big(I(D,K) - correction\big)$$

is a knot invariant. Further, it is a finite type k invariant. In fact, we get all finite type k invariants by varying W.

2. Configuration space integrals: notes

From here, it is not hard to see that there is an isomorphism

$$\mathcal{W}_k \xrightarrow[\cong]{\text{configuration space integrals}} \{\text{finite type } k \text{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}).$$

- This theorem is also called the *Fundamental Theorem of Finite Type (or Vassiliev) Invariants* and was first proved by Kontsevich using the famous *Kontsevich Integral*;
- Configuration space integrals are motivated by physics, more precisely by *Chern-Simons theory*;
- They are related to Lie algebras, Knizhnik-Zamolodchikov connection, etc.

Now let's talk about some generalizations and applications.

3(a). Generalization to knot spaces in dimension > 3

For $n > 3$, let

$$\mathcal{K}^n = \{\text{embeddings } \mathbb{R} \hookrightarrow \mathbb{R}^n \text{ fixed outside a compact set}\}$$

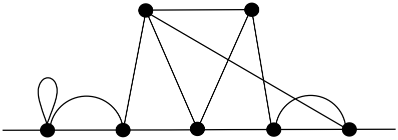
We are interested in computing

$$H_*(\mathcal{K}^n) \quad \text{and} \quad H^*(\mathcal{K}^n).$$

(H^0 and H_0 are now trivial, but higher (co)homology is very interesting.)

3(a). Generalization to knot spaces in dimension > 3

Take more general diagrams (*at least* trivalent), such as



For each $n \geq 3$, let

$$\mathcal{D}^n = \{\mathbb{R}\text{-vect. sp. gen'd by diagrams with valence} \geq 3\},$$

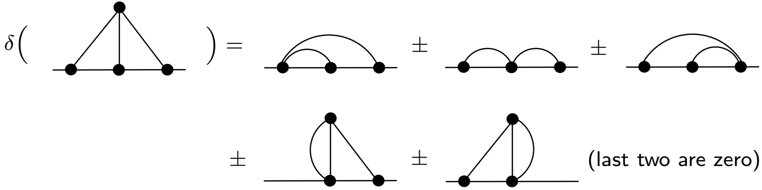
where diagrams are connected, vertices are labeled, no loops on off-segment vertices, edges are labeled or oriented (depending on parity of n). Mod out by diagrams with double edges and impose some sign relations.

Degree of $D \in \mathcal{D}$ is

$$\deg(D) = -2(\#\text{edges}) - 3(\#\text{off-segment vert.}) - (\#\text{segment vert.})$$

3(a). Generalization to knot spaces in dimension > 3

Coboundary δ is given by contracting non-chord and non-loop edges and segments, for example



Easy to see that δ raises degree by 1 and that $\delta^2 = 0$. Thus

$$\boxed{(\mathcal{D}^n, \delta) \text{ is a cochain complex.}}$$

Theorem (Cattaneo, Cotta-Ramusino, Longoni)

For $n > 3$, configuration space integrals give a cochain map

$$I_{\mathcal{K}} \colon (\mathcal{D}^n, \delta) \longrightarrow (\Omega^*(\mathcal{K}^n), d)$$

Conjecture

This map is a quasi-isomorphism.

3(a). Generalization to knot spaces in dimension > 3

For $n = 3$, one does not get a cochain map, but we can still try to see what happens on H^0. It turns out that

$$\mathrm{H}^0(\mathcal{D}^3) = \mathcal{TD} \quad \text{(trivalent diagrams)}$$

So kernel of δ in degree zero is defined by imposing the the STU and IHX relations. Thus we get a map (after identifying $\mathcal{TD}$ with its dual, the weight systems $\mathcal{W}$),

$$(\mathrm{H}^0(\mathcal{D}^3))^* = \mathcal{W} \longrightarrow \mathrm{H}^0(\mathcal{K}).$$

But we already know that the image of this map is precisely the finite type knot invariants.

3(b). Generalization to spaces of links and homotopy links

Related to long knot spaces in $\mathbb{R}^n$, $n \geq 3$, are

$$\mathcal{L}^n_m = \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\}$$
$$= \textit{space of long (string) links}$$
$$\mathcal{H}^n_m = \{\text{link maps} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\}$$
$$= \textit{space of homotopy long (string) links}$$
$$\mathcal{B}^n_m = \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n \text{ with positive derivative}\}$$
$$= \textit{space of pure braids}$$

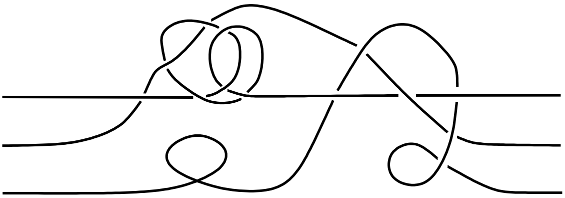
- All maps are standard outside a compact set;
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint.

Note:

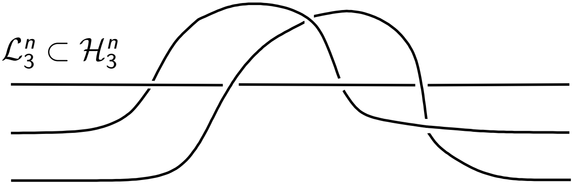
1. $\mathcal{B}^n_m \subset \mathcal{L}^n_m \subset \mathcal{H}^n_m$;
2. In $\pi_0(\mathcal{H}^n_m)$, can pass a strand through itself so this can be thought of as space of "links without knotting".

Space of long links and braids in $\mathbb{R}^3$

$H \in \mathcal{H}_3^n$



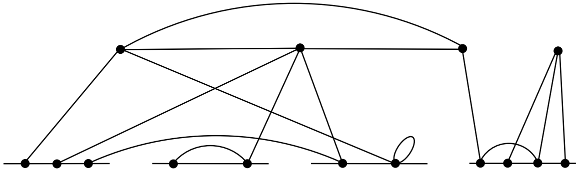
$L \in \mathcal{B}_3^n \subset \mathcal{L}_3^n \subset \mathcal{H}_3^n$



3(b). Generalization to spaces of links and homotopy links

Now generalize the diagram complex $\mathcal{D}^n$ to a complex $\mathcal{LD}^n_m$ and a subcomplex $\mathcal{HD}^n_m$.

$\mathcal{LD}^n_m$ is defined the same way as $\mathcal{D}^n$ except there are now m segments, e.g.

 $\in \mathcal{LD}^n_4$

$\mathcal{HD}^n_m$ is defined by imposing some relations.

3(b). Generalization to spaces of links and homotopy links

Theorem (Munson, V.)

There are integration maps $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{HD}^n_m & \stackrel{I_{\mathcal{H}}}{\hookrightarrow} & \Omega^*(\mathcal{H}^n_m) \\ \downarrow & & \downarrow \\ \mathcal{LD}^n_m & \xrightarrow{I_{\mathcal{L}}} & \Omega^*(\mathcal{L}^n_m) \end{array}$$

$I_{\mathcal{L}}$ is a cochain map for $n > 3$ and $I_{\mathcal{H}}$ is a cochain map for $n \geq 3$. Further, for $n = 3$, have isomorphisms

$$(\mathrm{H}^0(\mathcal{LD}^3_m))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{L}^3_m\} \in \mathrm{H}^0(\mathcal{L}^3_m)$$

$$(\mathrm{H}^0(\mathcal{HD}^3_m))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{H}^3_m\} \in \mathrm{H}^0(\mathcal{H}^3_m)$$

Conjecture

$I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are quasi-isomorphisms for $n > 3$ and $n \geq 3$, respectively.

3(b). Generalization to spaces of links and homotopy links

It is known that *Milnor invariants* of long homotopy links are finite type invariants. Thus get

Corollary

The map $I_{\mathcal{H}}$ provides configuration space integral expressions for Milnor invariants of $\mathcal{H}^3_m$.

Conjecture

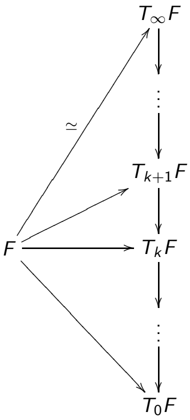
For closed homotopy links, there is a one-to-one correspondence, given by configuration space integrals, between trees and Milnor invariants.

Remarks:

1. It is somewhat surprising that configuration space integrals can be defined for homotopy links.
2. Can do all this for braids as well. One should be able to connect to work of T. Kohno on braids and Chen integrals.

3(c). Configuration space integrals and calculus of functors

Manifold calculus of functors is a theory that "approximates" certain kinds of functors F with values in Top. One gets a "Taylor tower" of functors that converges to F in some cases:



One such F is $\mathcal{K}^n$, $n > 3$. But the tower gives lots of information even for $n = 3$. More precisely,

3(c). Configuration space integrals and calculus of functors

Theorem (V.)

The Taylor tower for $\mathcal{K}$ classifies finite type invariants, i.e. there is a factorization

$$\mathcal{W}_k \xrightarrow[\cong]{\text{configuration space integrals}} \{\textit{fin. type } k \textit{ invariants}\} \subset \mathrm{H}^0(\mathcal{K})$$

$$\mathcal{W}_k \xrightarrow{\cong} \mathrm{H}^0(T_{2k}\mathcal{K}) \cong \mathrm{H}^0(T_{2k+1}\mathcal{K}) \xrightarrow{\cong} \{\textit{fin. type } k \textit{ invariants}\}$$

This places the separation conjecture into a homotopy-theoretic setting.

There is also a *multivariable* manifold functor calculus (Munson, V.) which applies to $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$. Same result as above is likely true for these spaces and their Taylor multi-towers.

Let $\mathcal{B}_n$ be the little balls operad in $\mathbb{R}^n$.

Theorem (Kontsevich, Tamarkin for $n = 2$)

For $n \geq 2$, there exists a chain of weak equivalences of operads of chain complexes

$$\mathrm{C}_*(\mathcal{B}_n; \mathbb{R}) \xleftarrow{\simeq} \textit{(some diagram complex)} \xrightarrow{\simeq} \mathrm{H}_*(\mathcal{B}_n; \mathbb{R})$$

In other words, $\mathcal{B}_n$ is (stably) formal over $\mathbb{R}$.

The right map (harder of the two) is given by configuration space integrals. These are different from what we've seen here since there is no knot present, but are essentially the same.

This result was used in the McClure-Smith proof of the Deligne Conjecture.

The connection to knots is the following.

3(d). Configuration space integrals and rat'l h'topy theory

Theorem (Lambrechts, Turchin, V.)

Formality of little balls + *Manifold calculus for $\mathcal{K}^n$*

$\rightsquigarrow$

Combinatorial description of rational H_* *and* π_* *of* $\mathcal{K}^n$, $n > 3$

Idea of proof.

Taylor tower has a cosimplicial variant which in turns comes with Bousfied-Kan homology and homotopy spectral sequences. Formality implies that these spectral sequences collapse at E^1, and this page is well-understood and can be described by diagrams. □

Conjecture

Same can be done for $\mathcal{L}^n_m$ and $\mathcal{H}^n_m$, although the latter is probably harder (we do not know if there is a convergence result for the Taylor tower of $\mathcal{H}^n_m$).

Thank you!