Configuration space integrals, operad formality, and the cohomology of knot and link spaces: Part I

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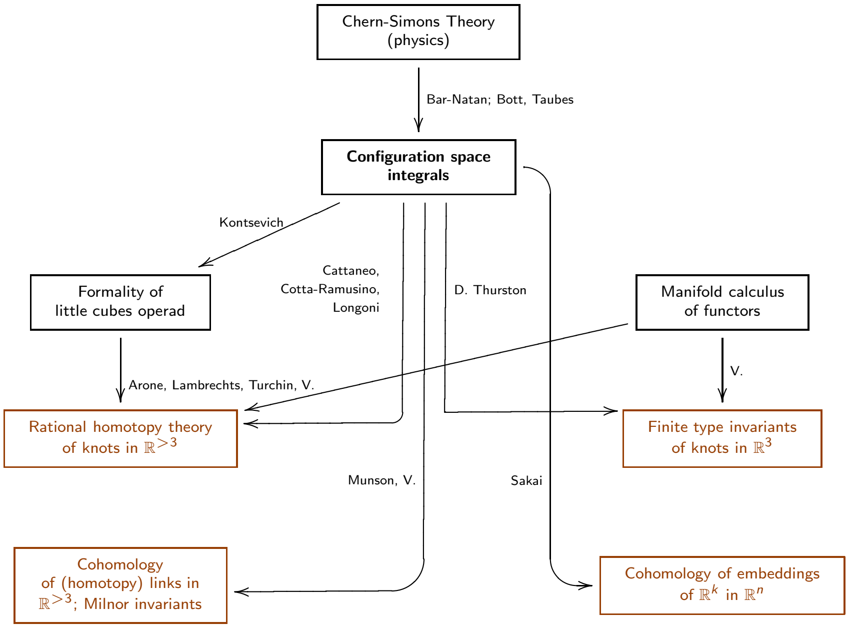
“Flow chart” of configuration space integrals



Outline of the talks

Talk 1: Configuration space integrals for knots

1. Embedding spaces and the special case of knots
2. Motivation for configuration space integrals: The linking number
3. Configuration space integrals for knots and finite type invariants
4. Generalization to cohomology of knot spaces in dimension > 3

Talk 2: Generalizations and applications

1. Calculus of functors for knots
   (a) Calculus of functors and finite type invariants
   (b) Configuration space integrals and rational homotopy theory
   (c) Calculus of functors and cohomology of knot spaces
2. Configuration space integrals for spaces of links
3. Configuration space integrals and multivariable calculus of functors for spaces of links
4. Configuration space integrals for embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$

1. Embedding spaces and the special case of knots

Definition

Let M and N be smooth manifolds. An *embedding* of M in N is an injective map $f\colon M \hookrightarrow N$ whose derivative is injective and which is a homeomorphism onto its image.

When M is compact, an embedding is an injective map with the injective derivative.

The set of all embeddings of M in N can be topologized so we get the *space of embeddings* $\mathrm{Emb}(M,N)$ (a special case of a *mapping space*).

For many M and N, this is a topologically interesting space, so we want to know

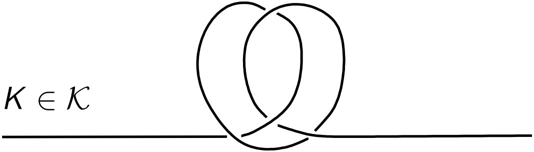
$$\pi_*(\mathrm{Emb}(M,N)),\quad \mathrm{H}_*(\mathrm{Emb}(M,N)),\quad \mathrm{H}^*(\mathrm{Emb}(M,N)).$$

Focus on $\mathrm{H}^*(\mathrm{Emb}(M,N);\mathbb{R})$ for the special case of *knots*.

1. Space of long knots in $\mathbb{R}^3$

$$\mathcal{K} = \{\text{embeddings } \mathbb{R} \hookrightarrow \mathbb{R}^3 \text{ fixed outside a compact set}\}$$
$$= \textit{space of long knots}$$

$K \in \mathcal{K}$



Classical knot theory is concerned with computing

$$\begin{aligned} \mathrm{H}_0(\mathcal{K}) &= \{\text{connected components of the space of knots}\} \\ &= \{\text{knot types}\} = \{\text{isotopy classes of knots}\} \\ \mathrm{H}^0(\mathcal{K}) &= \{\text{knot invariants } f\colon \mathrm{H}_0(\mathcal{K}) \to \mathbb{R}\}, \end{aligned}$$

1. Space of long knots in $\mathbb{R}^3$

If one knot can be deformed (isotoped) into another, an invariant $f \in \mathrm{H}^0(\mathcal{K})$ takes on the same value on those knots. But an invariant does not have to take on different values for different knots. In fact, we do not know if such an invariant or a class of invariants – a *complete* set of invariants that can tell all knots apart – exists.

Conjecture

The set of **finite type k invariants**, $k \geq 0$, *is a complete set of invariants.*

Finite type invariants have received much attention in the last 15 years:

- Motivated by physics (Chern-Simons Theory);
- Connected to Lie algebras, three-manifold topology, etc.

More about these later.

2. Linking number

Related to the space of knots $\mathcal{K}$ is

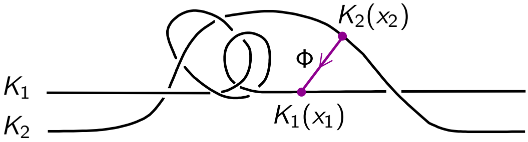
$$\mathcal{L}_2 = \{\text{embeddings } \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}^3, \text{ fixed outside a compact set}\}$$
$$= \textit{space of long (string) links of two components}$$

$$\text{Let } \operatorname{Conf}(k, \mathbb{R}^n) = \{(a_1, a_2, \ldots, a_k) \in \mathbb{R}^{kn} : a_i \neq a_j \text{ for } i \neq j\}$$
$$= \textit{configuration space of } k \textit{ points in } \mathbb{R}^n$$

Consider the maps Φ and π:

$$\Phi : \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 \xrightarrow{\text{evaluation}} \operatorname{Conf}(2, \mathbb{R}^3) \xrightarrow{\text{direction}} S^2$$

$$(x_1, x_2, L = (K_1, K_2)) \longmapsto (K_1(x_1), K_2(x_2)) \longmapsto \frac{K_2(x_2) - K_1(x_1)}{|K_2(x_2) - K_1(x_1)|}$$



$$\pi : \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 \xrightarrow{\text{projection}} \mathcal{L}_2 \quad \text{(trivial bundle)}$$

So have a diagram

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathcal{L}_2 & \xrightarrow{\Phi} & S^2 \\ \downarrow \pi & & \\ \mathcal{L}_2 & & \end{array}$$

which, on the complex of deRham cochains (differential forms), gives a diagram

$$\begin{array}{ccc} \Omega^*(\mathbb{R} \times \mathbb{R} \times \mathcal{L}_2) & \xleftarrow{\Phi^*} & \Omega^*(S^2) \\ \downarrow \pi_* & & \\ \Omega^{*-2}(\mathcal{L}_2) & & \end{array}$$

Here Φ^* is the usual pullback and π_* is *integration along the fiber*, or *pushforward* – a way to create forms on the base space of a bundle from forms on the total space, shifted by the dimension of the fiber.

1. Linking number

Let $sym_{S^2} \in \Omega^2(S^2)$ be the unit volume form on S^2, i.e.

$$sym_{S^2} = \frac{x\, dydz - y\, dxdz + z\, dxdy}{4\pi(x^2 + y^2 + z^2)^{3/2}}$$

Let $\alpha = \Phi^*(sym_{S^2})$. Then the linking number is

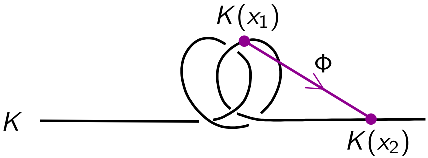
$$\boxed{Link(K_1, K_2) = \pi_*(\alpha) = \int_{\mathbb{R}\times\mathbb{R}} \alpha \in \Omega^0(\mathcal{L}_2)}$$

This is indeed a closed form, i.e. an element of $\mathrm{H}^0(\mathcal{L}_2)$, and hence an invariant of two-component links (this goes back to Gauss).

Now try to do the same, but for a single knot rather than a link.

2. Configuration space integrals: try to mimic $lk(K_1, K_2)$

The picture is



And the corresponding diagram is

$$\begin{array}{ccc} \mathrm{Conf}(2,\mathbb{R}) \times \mathcal{K} & \xrightarrow{\Phi} & S^2 \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{K} & & \end{array}$$

The first issue is that an integral over $\mathrm{Conf}(2,\mathbb{R})$ may not converge since this space is open. So need to compactify.

2. Configuration space integrals: Fulton-MacPherson compactification

Definition

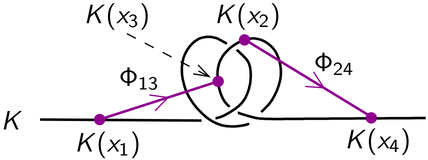
Let $\mathrm{Conf}[k, \mathbb{R}^n]$ be the *Fulton-MacPherson compactification* of $\mathrm{Conf}(k, \mathbb{R}^n)$.

Some properties:

- $\mathrm{Conf}[k, \mathbb{R}^n]$ is homotopy equivalent to $\mathrm{Conf}(k, \mathbb{R}^n)$;
- $\mathrm{Conf}[k, \mathbb{R}^n]$ is a manifold with corners;
- Boundary of $\mathrm{Conf}[k, \mathbb{R}^n]$ is characterized by points colliding with directions and relative rates of collisions kept track of;
- Codimension 1 boundary (important for Stokes' Theorem) given by points coming together at the same time (rather than in stages);
- $\mathrm{Conf}[k, \mathbb{R}]$ is the *associahedron*;
- Works for configurations in any manifold, not just $\mathbb{R}^n$.

2. Configuration space integrals: simplest case for knots

But, even after compactifying, we still do not get an invariant. The next case is that of four points and two directions:



The maps are

$$\begin{array}{ccc} \mathrm{Conf}[4,\mathbb{R}] \times \mathcal{K} & \xrightarrow{\Phi=\Phi_{13}\times\Phi_{24}} & S^2 \times S^2 \\ \big\downarrow \pi & & \\ \mathcal{K} & & \end{array}$$

Let $\alpha = \Phi^*(sym^2_{S^2})$. Since α and $\mathrm{Conf}[4,\mathbb{R}]$, the fiber of π, are both 4-dimensional, we get a 0-form

$I($  $, K) = \pi_*(\alpha) = \int_{\mathrm{Conf}[4,\mathbb{R}]} \alpha$

2. Configuration space integrals: simplest case for knots

So $I($$, K)$ is a 0-form, i.e. an element of $\Omega^0(\mathcal{K})$. But is it a closed form, that is, is it an element of $\mathrm{H}^0(\mathcal{K})$ – an invariant?

Want $dI($$, K) = 0$. Stokes' Theorem says that

$dI($$, K) = \pi_*(d\alpha) + (\partial\pi)_*(\alpha)$

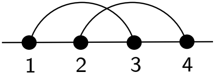
$$= (\partial\pi)_*(\alpha) \quad (\pi_*(d\alpha) = 0 \text{ since } sym_{S^2} \text{ is closed})$$

Here $(\partial\pi)_*(\alpha)$ is the pushforward along codimension one faces of $\mathrm{Conf}[4, \mathbb{R}]$.

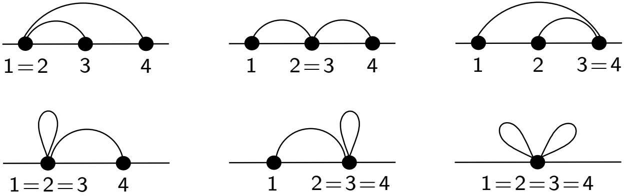
These faces can be represented by *diagrams* as follows.

2. Configuration space integrals: boundary diagrams

If there are four points moving on the knot, and two directions are kept track of as above, the diagram encoding this information is



Codimension one faces (collisions of points) are then encoded by diagrams obtained from the above one by contracting segments between points (this mimics collisions)

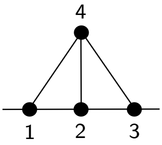


(Loop corresponds to the derivative map.)

It turns out that the integrals corresponding to the bottom three diagrams vanish, but not necessarily for the top three.

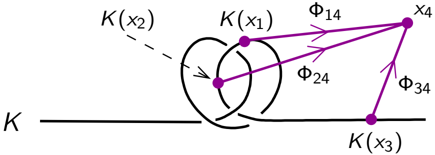
One way to resolve this: Look for another space to integrate over which has the same three faces and subtract the integrals.

The diagram that fits what we need is



since, when we contract edges to get $4=1$, $4=2$, and $4=3$, we get the same three relevant pictures as before (up to relabeling).

This suggests that we want a space of four configuration points in $\mathbb{R}^3$, three of which lie on a knot. In other words, we want the following picture:



2. Configuration space integrals: the fix

To make this precise, define pullback space

$$\begin{array}{ccc} \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \longrightarrow & \mathrm{Conf}[4,\mathbb{R}^3] \\ \downarrow & & \downarrow \scriptstyle{proj} \\ \mathrm{Conf}[3,\mathbb{R}]\times\mathcal{K} & \xrightarrow{eval} & \mathrm{Conf}[3,\mathbb{R}^3] \end{array}$$

Let

$$\Phi = \Phi_{14}\times\Phi_{24}\times\Phi_{34}\colon\ \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] \longrightarrow (S^2)^3$$

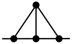
be the map giving the three directions as in the previous picture.

The natural map $\pi\colon\ \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3]\to\mathcal{K}$ is a bundle (Bott-Taubes) so the relevant maps are

$$\begin{array}{ccc} \mathrm{Conf}[3,1;\mathcal{K},\mathbb{R}^3] & \xrightarrow{\Phi} & (S^2)^3 \\ \downarrow \scriptstyle{\pi} & & \\ \mathcal{K} & & \end{array}$$

2. Configuration space integrals: the fix

Let $\alpha' = \Phi^*(sym^3_{S^2})$. This form can be integrated along the fiber $\mathrm{Conf}[3,1;K,\mathbb{R}^3]$ over K. Thus for each $K \in \mathcal{K}$, we get an integral

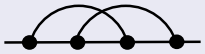
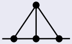
$I($  $, K) = \pi_*(\alpha') = \int_{\mathrm{Conf}[3,1;K,\mathbb{R}^3]} \alpha'$

It turns out that the boundary contributions for this integral are zero except for the three boundary pieces we care about. So we get

Theorem (Altschuler-Friedel, Bar-Natan)

The map

$$\mathcal{K} \longrightarrow \mathbb{R}$$

$K \longmapsto \Big(I($  $, K) - I($  $, K) \Big)$

is a knot invariant, i.e. an element of $\mathrm{H}^0(\mathcal{K})$. *Further, it is a finite type two invariant.*

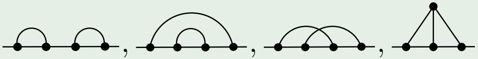
But there is no reason to stop at four configuration points...

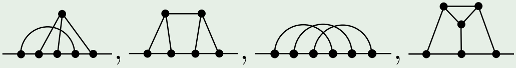
3. Configuration space integrals: general case

$\mathcal{TD}_k$ ={$\mathbb{R}$-vect. sp. gen'd by *trivalent diagrams* with $2k$ vertices, modulo *STU* and *IHX* relations}.

(STU and IHX are some relations on the vector space of diagrams.)

Example

$\mathcal{TD}_2 = \{$  $\}$

$\mathcal{TD}_3 = \{$  , etc.$\}$

Given $D \in \mathcal{TD}_k$ with p vertices on the segment and q off the segment, consider the pullback

$$\begin{array}{ccc} \mathrm{Conf}[p,q;\mathcal{K},\mathbb{R}^3] & \longrightarrow & \mathrm{Conf}[p+q,\mathbb{R}^3] \\ \downarrow & & \downarrow \mathit{proj} \\ \mathrm{Conf}[p,\mathbb{R}] \times \mathcal{K} & \xrightarrow{\mathit{eval}} & \mathrm{Conf}[p,\mathbb{R}^3] \end{array}$$

Then have map

$$\Phi: \text{Conf}[p, q; \mathcal{K}, \mathbb{R}^3] \longrightarrow (S^2)^e$$

where

- Φ is the product of the direction maps between pairs of configuration points corresponding to the edges of D, and
- e is the number of edges of D.

Let $\alpha = \Phi^*(sym^e_{S^2})$.

Then for each $K \in \mathcal{K}$, have integral

$$\boxed{I(D, K) = \pi_*(\alpha) = \int_{\text{Conf}[p,q;K,\mathbb{R}^3]} \alpha}$$

3. Configuration space integrals: general case

Let $\mathcal{W}_k = \mathcal{TD}_k^*$

Theorem (Bott-Taubes, D.Thurston)

For each $W \in \mathcal{W}_k$, the map

$$\mathcal{K} \longrightarrow \mathbb{R}$$
$$K \longmapsto \sum_{D \in \mathcal{TD}_k} W(D) I(D, K)$$

is a knot invariant[a]*. Further, it is a finite type k invariant. In fact, we get all finite type k invariants by varying W.*

[a] Slight lie.

This therefore gives a map

$$\mathcal{W}_k \longrightarrow \mathrm{H}^0(\mathcal{K})$$

which surjects onto finite type k invariants.

In fact, it is not hard to see that this is an isomorphism:

$$\boxed{\mathcal{W}_k \xrightarrow[\cong]{\text{configuration space integrals}} \{\text{finite type } k \text{ invariants}\} \subset \mathrm{H}^0(\mathcal{K}).}$$

This theorem is also called the *Fundamental Theorem of Finite Type (or Vassiliev) Invariants* and was first proved by Kontsevich using the famous *Kontsevich Integral*;

Now let's generalize to knots in $\mathbb{R}^n$, $n > 3$.

4. Generalization to knot spaces in dimension > 3

For $n > 3$, let

$$\mathcal{K}^n = \{\text{embeddings } \mathbb{R} \hookrightarrow \mathbb{R}^n \text{ fixed outside a compact set}\}$$

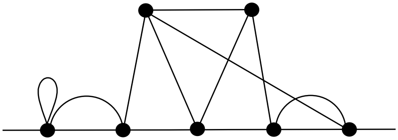
We are interested in understanding the topology of $\mathcal{K}^n$, i.e. we want to compute

$$\mathrm{H}_*(\mathcal{K}^n) \text{ and } \mathrm{H}^*(\mathcal{K}^n).$$

$\mathrm{H}^0(\mathcal{K}^n)$ and $\mathrm{H}_0(\mathcal{K}^n)$ are now trivial, but higher (co)homology is very interesting.

4. Generalization to knot spaces in dimension > 3

Take more general diagrams (*at least* trivalent), such as



For each $n \geq 3$, let

$$\mathcal{D}^n = \{\mathbb{R}\text{-vect. sp. gen'd by diagrams with valence} \geq 3\},$$

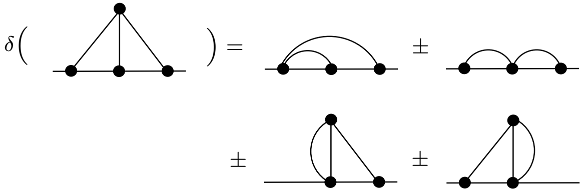
where diagrams are connected, vertices are labeled, no loops on off-segment vertices, edges are labeled or oriented (depending on parity of n). Mod out by diagrams with double edges and impose some sign relations.

Degree of $D \in \mathcal{D}$ is

$$\deg(D) = 2(\#\text{edges}) - 3(\#\text{off-segment vert.}) - (\#\text{segment vert.})$$

4. Generalization to knot spaces in dimension > 3

Coboundary δ is given by contracting non-chord and non-loop edges and segments, for example



(last two are zero)

Easy to see that δ raises degree by 1 and that $\delta^2 = 0$. Thus

$$\boxed{(\mathcal{D}^n, \delta) \text{ is a cochain complex.}}$$

4. Generalization to knot spaces in dimension > 3

For each $D \in \mathcal{D}^n$ and $K \in \mathcal{K}^n$, we can still set up the integral $I(D, K)$ as before. The only difference is that we will not necessarily get a form in degree zero but in some degree of $\Omega^*(\mathcal{K}^n)$.

Theorem (Cattaneo, Cotta-Ramusino, Longoni)

For $n > 3$, configuration space integrals give a cochain map

$$I_{\mathcal{K}}\colon (\mathcal{D}^n, \delta) \longrightarrow (\Omega^*(\mathcal{K}^n), d).$$

Corollary

The knot space $\mathcal{K}^n$, $n > 3$, has nontrivial cohomology beyond arbitrarily high dimension.

Conjecture

This map is a quasi-isomorphism.

4. Generalization to knot spaces in dimension > 3

This is compatible with what we already did in the case of classical knots $\mathcal{K} = \mathcal{K}^3$:

For $n = 3$, one does not get a cochain map in all degrees, but in degree zero the map can be modified so that it does commute with the differential. So we can see what happens on H^0. It turns out that

$$\mathrm{H}^0(\mathcal{D}^3) = \mathcal{TD} \quad \text{(trivalent diagrams)}$$

So kernel of δ in degree zero is defined by imposing the the STU and IHX relations. Thus we get a map (after identifying $\mathcal{TD}$ with its dual, the weight systems $\mathcal{W}$),

$$(\mathrm{H}^0(\mathcal{D}^3))^* = \mathcal{W} \longrightarrow \mathrm{H}^0(\mathcal{K}).$$

But we already know that the image of this map is precisely the finite type knot invariants.

Preview of second talk

Next time, we'll talk about

- Manifold calculus of functors and a homotopy-theoretic framework for finite type invariants;
- Generalization to links and homotopy links and interesting connection to Milnor invariants (and lots of open questions);
- Configuration space integrals and operad formality, which leads to;
- A combinatorial description of the rational homology and homotopy of the space of knots in $\mathbb{R} > 3$;
- Brief mention of the generalization to embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$.