Configuration space integrals, operad formality, and the cohomology of knot and link spaces: Part II

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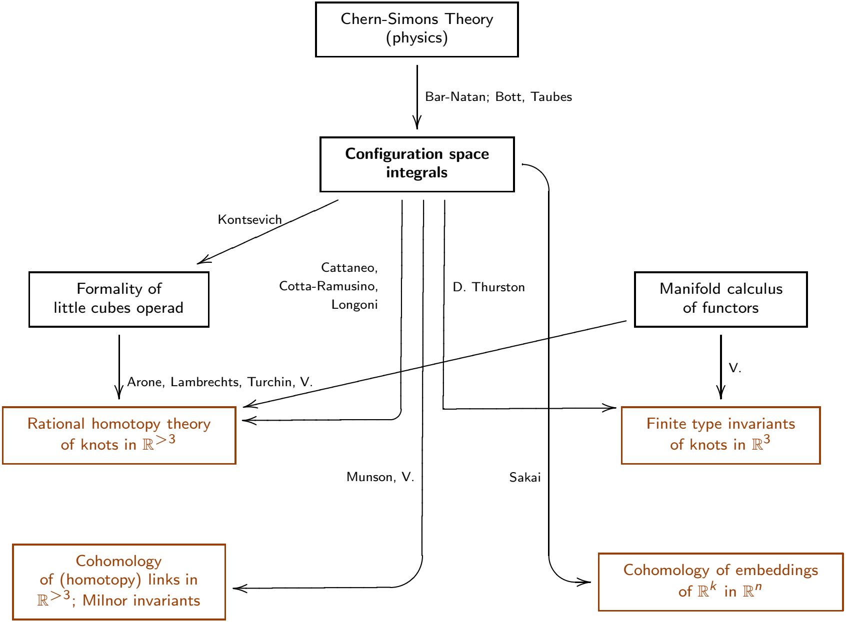
“Flow chart” of configuration space integrals



Outline of the talks

Talk 1: Configuration space integrals for knots

1. Embedding spaces and the special case of knots
2. Motivation for configuration space integrals: The linking number
3. Configuration space integrals for knots and finite type invariants
4. Generalization to cohomology of knot spaces in dimension > 3

Talk 2: Generalizations and applications

1. Calculus of functors for knots
   (a) Calculus of functors and finite type invariants
   (b) Configuration space integrals and rational homotopy theory
   (c) Calculus of functors and cohomology of knot spaces
2. Configuration space integrals for spaces of links
3. Configuration space integrals and multivariable calculus of functors for spaces of links
4. Configuration space integrals for embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$

Review of the first talk

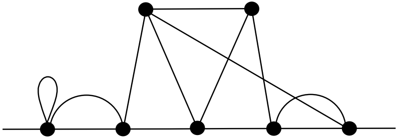
For $n = 3$, have

$$\mathcal{K}^n = \{\text{embeddings } \mathbb{R} \hookrightarrow \mathbb{R}^n\}$$
$$= \textit{space of long knots},$$

a diagram complex $(\mathcal{D}^n, \delta)$, and a cochain map

$$I_{\mathcal{K}} \colon (\mathcal{D}^n, \delta) \longrightarrow (\Omega^*(\mathcal{K}^n), d)$$

This map is constructed as follows: Given a diagram $D \in \mathcal{D}^n$, e.g.



with p vertices on the segment and q off it, we have a space $\mathrm{Conf}[p, q; \mathcal{K}, \mathbb{R}^3]$ on which we can produce a differential form α according to the edges of D.

Review of the first talk

This form α can then be pushed forward to $\mathcal{K}^n$ via the bundle map

$$\pi\colon \text{Conf}[p,q;\mathcal{K},\mathbb{R}^3] \longrightarrow \mathcal{K}^n.$$

So $I_{\mathcal{K}}(D) = \pi_*(\alpha)$ and this is a cochain map (the differential δ on $\mathcal{D}^n$ contracts edges).

In the case $n = 3$, we do not get a cochain map, but we can still examine what happens in degree zero, where

$$\mathrm{H}^0(\mathcal{D}^3) = \mathcal{TD}$$

($\mathcal{TD}$ is trivalent diagrams modulo STU and IHX relations). We then get that configuration space integrals give an isomorphism between the dual of $\mathcal{TD}$, denoted by $\mathcal{W}$, and finite type invariants:

$$\mathcal{W} \xrightarrow{\quad\cong\quad} \{\text{finite type invariants}\} \subset \mathrm{H}^0(\mathcal{K}).$$

($\mathcal{W}$ is graded so that we in fact have isomorphisms

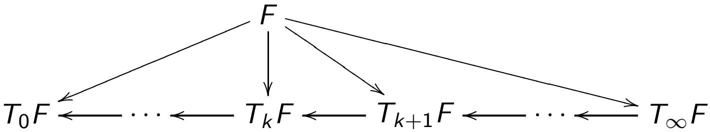
$$\mathcal{W}_k \xrightarrow{\cong} \{\text{finite type } k \text{ invariants}\}.)$$

1. Calculus of functors

Calculus of functors is a theory that aims to "approximate" functors in algebra in topology much like the Taylor polynomials approximate ordinary smooth real or complex-valued functions. In general, given a functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

this theory gives a "Taylor tower" of approximating functors:



Depending on F, this tower might *converge*, i.e. there might an equivalence, for all $X \in \mathcal{C}$,

$$F(X) \simeq T_\infty F(X).$$

1. Calculus of functors

There are currently three varieties of functor calculus:

- Homotopy calculus
- Orthogonal calculus
- Manifold calculus

Each is designed to study different kinds of functors. Here we are interested in manifold calculus:

Given a smooth manifold M, its open subsets form a category $\mathcal{O}(M)$ with inclusions maps as morphisms. Manifold calculus then studies contravariant functors

$$F\colon \mathcal{O}(M) \longrightarrow \mathrm{Top}$$

The main example of such a functor is the space of embeddings $\mathrm{Emb}(M, N)$:

1. Calculus of functors

$$\begin{aligned} \operatorname{Emb}(-,N)\colon \mathcal{O}(M) &\longrightarrow \mathrm{Top} \\ O &\longmapsto \operatorname{Emb}(O,N) \end{aligned}$$

This is contravariant since, given an inclusion $O_1 \hookrightarrow O_2$ of open subsets of M, there is a restriction

$$\operatorname{Emb}(O_2, N) \to \operatorname{Emb}(O_1, N).$$

Manifold calculus thus applies to the functor $\operatorname{Emb}(-,N)$ and we get a Taylor tower

$$\begin{array}{c} \operatorname{Emb}(-,N) \\ \swarrow \qquad \downarrow \qquad \searrow \qquad \searrow \\ T_0 \operatorname{Emb}(-,N) \leftarrow \cdots \leftarrow T_k \operatorname{Emb}(-,N) \leftarrow T_{k+1} \operatorname{Emb}(-,N) \leftarrow \cdots \leftarrow T_\infty \operatorname{Emb}(-,N) \end{array}$$

Theorem (Goodwillie-Klein-Weiss)

The Taylor tower for $\mathrm{Emb}(-, N)$ *converges under certain dimensional assumptions. Namely, given* $O \in \mathcal{O}(M)$, *the map*

$$\mathrm{Emb}(O, N) \longrightarrow T_\infty \mathrm{Emb}(O, N)$$

induces isomorphisms

- *on* π_* *if* $dim(M) + 3 \leq dim(N)$, *and*
- *on* H_* *and* H^* *if* $4 dim(M) \leq dim(N)$.

In practice, we set $O = M$ to extract information about the space of embeddings of the entire manifold (so functoriality is used in proving the statement, but we then specialize).

One example of $\mathrm{Emb}(M, N)$ is $\mathcal{K}^n$. Note that the above theorem says nothing about $\mathcal{K}^3 = \mathcal{K}$.

Let's look at the construction of $T_k\mathcal{K}^n$, $n \geq 3$:

1. Calculus of functors for knots

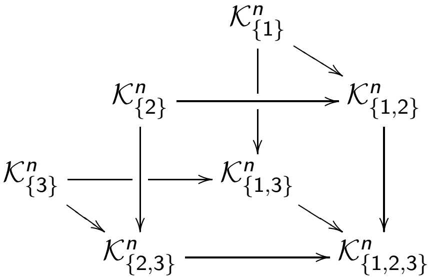
Let $I_0, \ldots, I_{k+1}$ be disjoint subintervals of $\mathbb{R}$ and

$$\emptyset \neq S \subseteq \{1, \ldots, k+1\}.$$

Then let

$$\mathcal{K}_S^n = \mathrm{Emb}(\mathbb{R} \setminus \bigcup_{i \in S} I_i,\ \mathbb{R}^n) = \text{space of “punctured knots”}$$

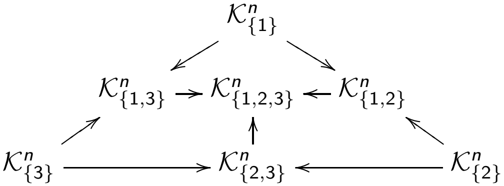
Have restriction maps $\mathcal{K}_S^n \to \mathcal{K}_{S \cup \{i\}}^n$ (punch another hole) which form a (sub)cubical diagram of punctured knots. E.g., when $k = 2$, we get



1. Calculus of functors for knots

$$\boxed{T_k\mathcal{K}^n = \operatorname{holim}_{\emptyset \neq S \subseteq \{1,..,k+1\}} \mathcal{K}^n_S.}$$

Not hard to see what this homotopy limit is: The diagram from the previous slide can be redrawn as



Then a point in $T_2\mathcal{K}^n$ is

- A point in each $\mathcal{K}^n_{\{i\}}$ (once-punctured knot);
- A path in each $\mathcal{K}^n_{\{i,j\}}$ (isotopy of a twice-punctured knot) ;
- A two-parameter path in $\mathcal{K}^n_{\{1,2,3\}}$ (two-parameter isotopy of a thrice-punctured knot); and
- Everything is compatible with the restriction maps.

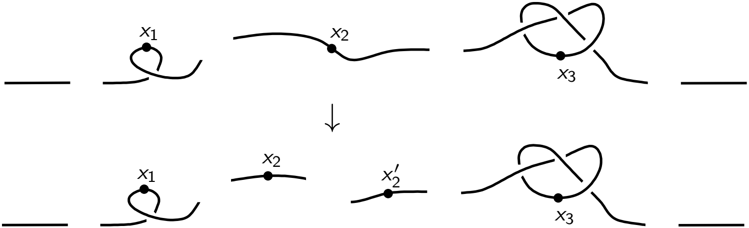
1. Calculus of functors for knots

Where is the connection to configuration spaces?

$$\mathcal{K}^n_S \simeq \mathrm{Conf}(|S|-1, \mathbb{R}^n)$$

(Only true when we're in the fiber of the inclusion Emb $\hookrightarrow$ Imm, which we usually are.)

The restriction maps "add a point":



(This is made precise by a *cosimplicial space*

$$(K^n)^\bullet = \left(\mathrm{Conf}[0, \mathbb{R}^n] \rightrightarrows \mathrm{Conf}[1, \mathbb{R}^n] \rightrightarrows \mathrm{Conf}[2, \mathbb{R}^n] \cdots \right),$$

where maps are doubling (diagonal) and forgetting. Partial totalizations of $(K^n)^\bullet$ are precisely the stages of the Taylor tower.)

1(a). Calculus of functors and finite type invariants

Theorem (V.)

The Taylor tower for $\mathcal{K}$ classifies finite type invariants.

Idea of proof.

Show that configuration space integrals factor through the Taylor tower and that all the maps are isomorphisms:

$$\mathcal{W}_k \xrightarrow[\cong]{\text{configuration space integrals}} \{\text{fin. type } k \text{ invariants}\} \subset \mathrm{H}^0(\mathcal{K})$$

$$\mathcal{W}_k \xrightarrow{\cong} \mathrm{H}^0(T_{2k}\mathcal{K}) \cong \mathrm{H}^0(T_{2k+1}\mathcal{K}) \xrightarrow{\cong} \{\text{fin. type } k \text{ invariants}\}$$

□

This places the conjecture that finite type invariants separate knots into a homotopy-theoretic setting.

1(b). Configuration space integrals and rat'l h'topy theory

Before we can talk about calculus of functors and $\mathcal{K}^n$, $n > 3$, we need a digression to rational homotopy theory:

Let $\mathcal{B}_n$ be the little n-discs operad (i.e. little balls in $\mathbb{R}^n$).

Theorem (Kontsevich, Tamarkin for $n = 2$)

For $n \geq 2$, there exists a chain of weak equivalences of operads of chain complexes

$$\mathrm{C}_*(\mathcal{B}_n; \mathbb{R}) \overset{\simeq}{\longleftarrow} \textit{(some diagram complex)} \overset{\simeq}{\longrightarrow} \mathrm{H}_*(\mathcal{B}_n; \mathbb{R})$$

In other words, $\mathcal{B}_n$ is (stably) formal over $\mathbb{R}$.

The left map (harder of the two) is given by configuration space integrals. It is essentially the same as the map $I_{\mathcal{K}}$ we saw before:

- The diagram complex in the middle is essentially $\mathcal{D}^n$;
- Integration setup is exactly the same except the integration takes place along the bundle $\mathrm{Conf}[p+q, \mathbb{R}^n] \to \mathrm{Conf}[p, \mathbb{R}^n]$ (rather than $\mathrm{Conf}[p, q; \mathcal{K}^n, \mathbb{R}^n] \to \mathcal{K}^n$).

1(b). Configuration space integrals and rat'l h'topy theory

Some notes on this important result:

- Used in the McClure-Smith proof of the Deligne Conjecture;
- Used in proof of Kontsevich's deformation quantization theorem (Tamarkin);
- Extends to *stable* formality (Lambrechts, V.), which leads to;
- Results about the homology of $\mathrm{Emb}(M, \mathbb{R}^n)$ (Arone, Lambrechts, V.);
- Extends to framed and cyclic 2-discs operads (Giansiracusa, Salvatore);
- Framed version used in construction of BV-algebra structures for topological conformal field theories (Gálvez-Carrillo, Tonks, Vallette).

For us, formality is relevant for getting at the cohomology of $\mathcal{K}^n$, $n > 3$:

1(c). Calculus of functors and cohomology of knot spaces

Theorem (Lambrechts, Turchin, V.)

The rational homology of $\overline{\mathcal{K}^n} = \mathrm{hofiber}(\mathcal{K}^n \hookrightarrow \mathrm{Imm}(\mathbb{R}, \mathbb{R}^n))$ *(this is a space very closely related to* $\mathcal{K}^n$*),* $n > 3$*, is the Hochschild homology of the little n-discs operad.*

An easier thing to remember:

This theorem says that $\mathrm{H}_*(\overline{\mathcal{K}^n}; \mathbb{Q})$ is built out of $\mathrm{H}_*(\mathrm{Conf}[p, \mathbb{R}^n]; \mathbb{Q})$, $p \geq 0$. Further, $\mathrm{H}_*(\mathrm{Conf}[p, \mathbb{R}^n]; \mathbb{Q})$ can be expressed with *chord diagrams* (special case of trivalent diagrams we've seen), so we get a combinatorial description of $\mathrm{H}_*(\overline{\mathcal{K}^n}; \mathbb{Q})$, $n > 3$.

Idea of proof.

Taylor tower for $\overline{\mathcal{K}^n}$ has a cosimplicial variant which comes with Bousfied-Kan homology spectral sequence. Formality implies that this spectral sequence collapses at E^1, and this page consists of the homology of configuration spaces. □

(Similar results true for $\pi_*(\overline{\mathcal{K}^n}) \otimes \mathbb{Q}$ (with Arone).)

2. Configuration space integrals for spaces of links

Let $n \geq 3$ and $m \geq 1$. Related to $\mathcal{K}^n$ are

$$\begin{aligned} \mathcal{L}^n_m &= \{\text{embeddings} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\ &= \textit{space of long (string) links} \\ \mathcal{H}^n_m &= \{\text{link maps} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\ &= \textit{space of homotopy long (string) links} \\ \mathcal{B}^n_m &= \{\text{embeddings with positive derivative} \quad \sqcup_m \mathbb{R} \hookrightarrow \mathbb{R}^n\} \\ &= \textit{space of pure braids} \end{aligned}$$

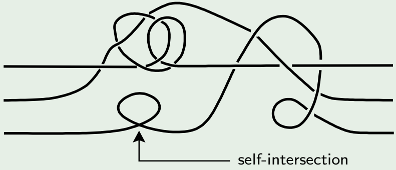
- All maps are standard outside a compact set;
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint.

(As with knots, we in practicework with the homotopy fiber of the inclusion embeddings $\hookrightarrow$ immersions for $\mathcal{L}^n_m$ and $\mathcal{B}^n_m$.)

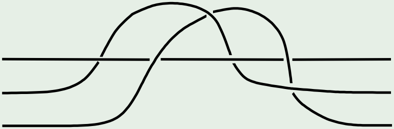
- $\mathcal{B}^n_m \subset \mathcal{L}^n_m \subset \mathcal{H}^n_m$;
- In $\pi_0(\mathcal{H}^n_m)$, can pass a strand through itself so this can be thought of as space of "links without knotting".

Example

$H \in \mathcal{H}^n_3$



$L \in \mathcal{B}^n_3 \subset \mathcal{L}^n_3 \subset \mathcal{H}^n_3$



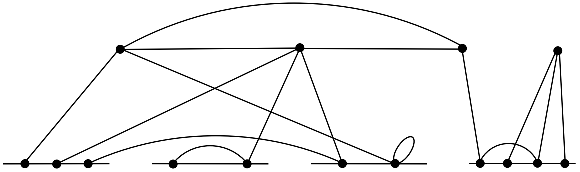
2. Configuration space integrals for spaces of links

Recall the cochain map

$$I_{\mathcal{K}} \colon \mathcal{D}^n \longrightarrow \Omega^*(\mathcal{K}^n)$$

Generalize the diagram complex $\mathcal{D}^n$ to a complex $\mathcal{LD}^n_m$ and a subcomplex $\mathcal{HD}^n_m$.

$\mathcal{LD}^n_m$ is defined the same way as $\mathcal{D}^n$ except there are now m segments, e.g.

 $\in \mathcal{LD}^n_4$

$\mathcal{HD}^n_m$ is defined by imposing: If there exists a path between distinct vertices on a given segment, then it must pass through a vertex on another segment.

2. Configuration space integrals for spaces of links

Theorem (Munson, V.)

There are integration maps $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{HD}^n_m & \xrightarrow{I_{\mathcal{H}}} & \Omega^*(\mathcal{H}^n_m) \\ \downarrow & & \downarrow \\ \mathcal{LD}^n_m & \xrightarrow{I_{\mathcal{L}}} & \Omega^*(\mathcal{L}^n_m) \end{array}$$

$I_{\mathcal{L}}$ is a cochain map for $n > 3$ and $I_{\mathcal{H}}$ is a cochain map for $n \geq 3$. Further, for $n = 3$, we have isomorphisms

$$(\mathrm{H}^0(\mathcal{LD}^3_m))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{L}^3_m\} \in \mathrm{H}^0(\mathcal{L}^3_m)$$
$$(\mathrm{H}^0(\mathcal{HD}^3_m))^* \xrightarrow{\cong} \{\textit{fin. type inv's of } \mathcal{H}^3_m\} \in \mathrm{H}^0(\mathcal{H}^3_m)$$

Conjecture

$I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are quasi-isomorphisms for $n > 3$ and $n \geq 3$, respectively.

2. Configuration space integrals for spaces of links

It is known that *Milnor invariants* of long homotopy links are finite type invariants. Thus get

Corollary

The map $I_{\mathcal{H}}$ provides configuration space integral expressions for Milnor invariants of $\mathcal{H}^3_m$.

Conjecture

For closed homotopy links, there is a one-to-one correspondence, given by configuration space integrals, between trees and Milnor invariants.

Remarks:

1. It is somewhat surprising that configuration space integrals can be defined for homotopy links.
2. Can do all this for braids as well. One should be able to connect to work of T. Kohno on braids and Chen integrals.

3. Multivariable calculus and links (two variables

If $M = P \coprod Q$, can apply two-variable calculus (Munson-V.) for contravariant functors F on $\mathcal{O}(P) \times \mathcal{O}(Q)$ (rather than on $\mathcal{O}(P \coprod Q)$). Get bitower

$$
\begin{array}{ccccccc}
T_{0,\infty}F(-,-) & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & T_{\infty,\infty}F(-,-) \\
\downarrow & & & & & & \downarrow \\
\vdots & & & & & & \vdots \\
\downarrow & & & & & & \downarrow \\
T_{0,1}F(-,-) & \longleftarrow & T_{1,1}F(-,-) & \longleftarrow & \cdots & \longleftarrow & T_{\infty,1}F(-,-) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_{0,0}F(-,-) & \longleftarrow & T_{1,0}F(-,-) & \longleftarrow & \cdots & \longleftarrow & T_{\infty,0}F(-,-)
\end{array}
$$

- Connection to single-variable calculus: $T_k F = \operatorname{holim}_{k_1+k_2 \leq k} T_{k_1,k_2} F$;
- Have same convergence result as before: For $F = \operatorname{Emb}(P \coprod Q, N)$ and same codimension assumptions, the bitower converges.

3. Multivariable calculus and links (two variables)

This construction applies to $\mathcal{L}_m^n$, $\mathcal{H}_m^n$, and $\mathcal{B}_m^n$. The convergence thus holds for $\mathcal{L}_m^n$ and $\mathcal{B}_m^n$, $n > 3$, but we do not know anything about $\mathcal{H}_m^n$!

To construct $T_{k_1,k_2}\mathcal{L}_2^n$:

Let $I_0, \ldots, I_{k_1+1}$ be disjoint intervals in $\mathbb{R}$. Same for $J_0, \ldots, J_{k_2+1}$. Then

Definition

$$T_{k_1,k_2}\mathcal{L}_2^n = \operatorname{holim}_{\substack{\emptyset \neq S_1 \subseteq \{1,\ldots,k_1\} \\ \emptyset \neq S_2 \subseteq \{1,\ldots,k_2\}}} \operatorname{Emb}\left((\mathbb{R} \backslash \bigcup_{i \in S_1} I_i) \coprod (\mathbb{R} \backslash \bigcup_{j \in S_j} J_j) \, , \, \mathbb{R}^n \right).$$

These "punctured links" still look like configuration spaces but the most important ones are spaces $\operatorname{Conf}[2k, \mathbb{R}^n]$.

(Get a bicosimplicial space of configuration spaces $\operatorname{Conf}[k_1 + k_2, \mathbb{R}^n]$ and look at the Bousfield-Kan spectral sequence of the diagonal cosimplicial space consisting of $\operatorname{Conf}[2k, \mathbb{R}^n]$.)

3. Multivariable calculus and links (two variables)

Setup is same for $\mathcal{H}_2^n$ and $\mathcal{B}_2^n$, except:

- For $\mathcal{H}_2^n$, the relevant spaces are

  $$\mathrm{Conf}[k, k; \mathbb{R}^n] = \text{compactification of } \{(x_1, ..., x_k, y_1, ..., y_k) \in (\mathbb{R}^n)^{2k} : x_i \neq y_j\}$$

  This is a kind of a compactified "partial configuration space" or a complement of a subspace arrangement.

- For $\mathcal{B}_2^n$, the relevant spaces are

  $$(\mathrm{Conf}[2, \mathbb{R}^{n-1}])^k$$

  (This gives the standard cosimplicial model for $\Omega\,\mathrm{Conf}(2, \mathbb{R}^{n-1})$, which is exactly what braids are.)

3. Multivariable calculus and links

Generalization to m-component links is straightforward: Get, for $n \geq 3$,

- m-dimensional Taylor towers for $\mathcal{L}_m^n$, $\mathcal{H}_m^n$, and $\mathcal{B}_m^n$ (and the corresponding m-cosimplicial models);
- Relevant spaces:
  - For $\mathcal{L}_m^n$, get $\mathrm{Conf}[km, \mathbb{R}^n]$, $k \geq 0$;
  - For $\mathcal{H}_m^n$, get $\mathrm{Conf}[k, k, ..., k; \mathbb{R}^n]$, $k \geq 0$; and
  - For $\mathcal{B}_m^n$, get $(\mathrm{Conf}[m, \mathbb{R}^{n-1}])^k$, $k \geq 0$

So how many of the results we had for knots carry over to these spaces of links?

3. Multivariable calculus and links

Very likely true (need to work out details):

Since Taylor multi-tower converges for embeddings for $n > 3$, we can obtain a combinatorial description, via chord diagrams, of rational homology of $\mathcal{L}_m^n$ and $\mathcal{B}_m^n$ from configuration spaces. Formality would probably again play a role here.

Likely true (needs a lot of work):

Show the same for $\mathcal{H}_m^n$. This is a lot harder since we no longer have convergence of the Taylor multi-tower. It is also not clear if "partial configuration spaces" are formal or how to prove this.

Very likely true (need to work out details):

Taylor multi-towers classify finite type invariants of $\mathcal{L}_m^3$, $\mathcal{H}_m^3$, and $\mathcal{B}_m^3$. Further consequence: Recalling that Milnor invariants of $\mathcal{H}_m^3$ are finite type, this would place these invariants in the context of manifold calculus of functors.

3. Functor calculus and knots/links: Further work

So the main point is

Taylor towers contain information about topology of knots and links and configuration space integrals help us understand it.

To do, in addition to completing the likely and very likely things from previous slide:

- Reprove, in the setting of Taylor towers, that finite type invariants separate braids (Kohno, Bar-Natan) and homotopy string links (Habegger-Lin).
- See if this helps in proving the same result for knots and links.
- Use configuration space integrals and functor calculus to get more information about the entire cohomology in the case $n = 3$, for all knot and link spaces (so far we only have information about degree zero).
- Generalize Milnor invariants (which are finite type) to homotopy links of spheres (or planes) in any dimension and connect with work of Koschorke. Show, using manifold calculus, that these generalizations suffice for separation of link maps of spheres.

4. Generalization to spaces of embeddings of $\mathbb{R}^k$ in $\mathbb{R}^n$

K. Sakai has recently done a lot of work on configuration space integrals:

- Produces a cohomology class of $\mathcal{K}^3$ in degree one that is related to the Casson invariant using integrals;
- (with Watanabe) There is a diagram complex $\mathcal{D}^{k,n}$ and a linear map

  $$\mathcal{D}^{k,n} \longrightarrow \mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$$

  which is a cochain map for some subcomplexes of $\mathcal{D}^{k,n}$ and certain parity conditions on k and n. Also have some non-triviality results;
- New interpretation of the Haefliger invariant for $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^n)$ for some k and n.

It would be nice to connect this work to functor calculus as well.

Thank you!