Transparencies for AMS meeting, Houston 5/04

ON THE RATIONAL HOMOTOPY TYPE OF SPACES OF KNOTS

(CODIM ≥ 3)

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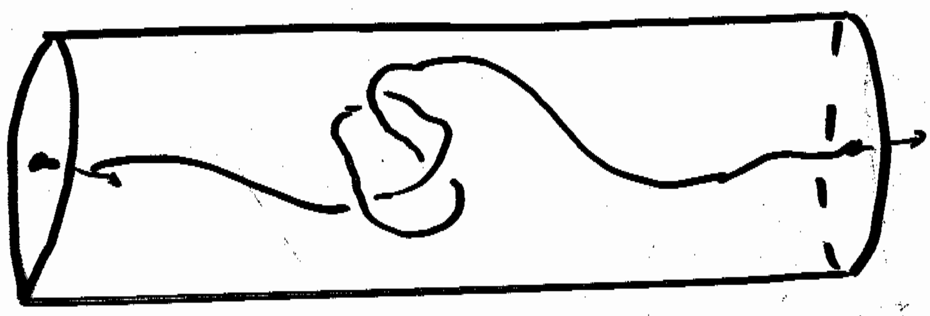
$\mathcal{K} = \mathrm{Emb}(I, I \times D^{n-1})$,
fixed on ∂, $n > 3$.
("string knots", or "long knots")



<u>Goal</u>: Find $H^*(\mathcal{K}; \mathbb{Q})$
$\pi_*(\mathcal{K}) \otimes \mathbb{Q}$

<u>Better</u>: Find a differential graded algebra (DGA) (A, d) and quasi-isomorphisms

$$(A, d) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} (\Omega^*(\mathcal{K}), d)$$

At least two approaches so far:

① Vassiliev / Kontsevich / Turchin

- Vassiliev sets up a spectral sequence converging to $H^*(\mathcal{K};\mathbb{Q})$
- Kontsevich conjectures collapse at E_2

② Bott-Taubes / Cattaneo
Cotta-Ramusino
Longoni

- use Bott-Taubes integrals to produce cohomology classes

We use

③ Goodwillie calculus of functors

Taylor tower for $\mathcal{K}$

Weiss sets up a tower for studying $\mathrm{Emb}(M, N)$:

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_{r+1} \arrow[d] \\
\mathrm{Emb}(M,N) \arrow[ur] \arrow[r] \arrow[dr] & T_r \arrow[d] \\
 & \vdots
\end{tikzcd}
$$

Thm (Goodwillie - Klein - Weiss):

If $\dim N - \dim M > 2$, this tower converges.

(In particular, $\mathcal{K} \longrightarrow T_r$ is $r(n-3)$ - connected)

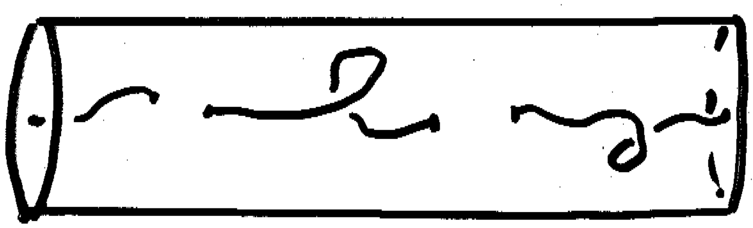
To construct T_3 for $\mathcal{K}$:

Let A_1, A_2, A_3 be subintervals of I

For $\emptyset \neq S \subseteq \{1,2,3\}$, let

$$E_S = \mathrm{Emb}\left(I \setminus \bigcup_{i \in S} A_i \,,\; I \times D^{n-1}\right)$$

= "punctured knots"



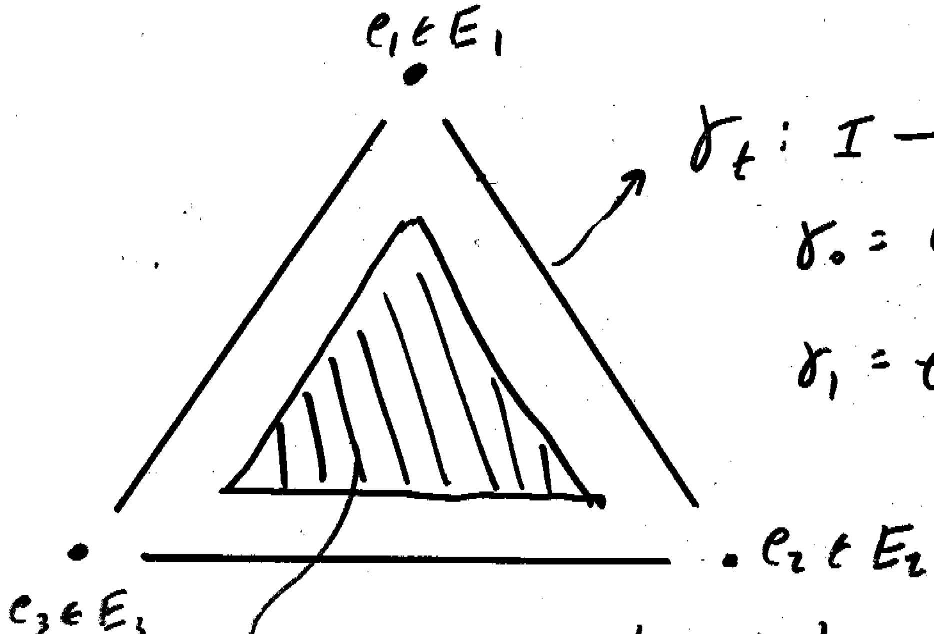
There are restrictions which commute:

$$\begin{tikzcd} E_S \arrow[r] \arrow[d] & E_{S \cup \{i\}} \arrow[d] \\ E_{S \cup \{j\}} \arrow[r] & E_{S \cup \{i,j\}} \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{K} \arrow[rr] \arrow[dr] \arrow[dd] & & E_1 \arrow[dr] \arrow[dd] & \\
& E_2 \arrow[rr] \arrow[dd] & & E_{12} \arrow[dd] \\
E_3 \arrow[rr] \arrow[dr] & & E_{13} \arrow[dr] & \\
& E_{23} \arrow[rr] & & E_{123}
\end{tikzcd}
$$

$$
T_3 = \operatorname{holim} \left(\begin{tikzcd}
& & E_1 \arrow[dl] \arrow[dr] & & \\
& E_{13} \arrow[dr] & & E_{12} \arrow[dl] & \\
& & E_{123} & & \\
E_3 \arrow[uur] \arrow[rr] & & E_{23} \arrow[u] & & E_2 \arrow[uul] \arrow[ll]
\end{tikzcd} \right)
$$

$=$ {



$\gamma_t : I \longrightarrow E_{12}$

$\gamma_0 = e_1 |_{E_{12}}$

$\gamma_1 = e_2 |_{E_{12}}$

2-parameter isotopy in E_{123}, compatible

}

In general,

$$\boxed{T_r = \operatorname*{holim}_{\emptyset \neq S \subseteq \{1,\dots,n\}} \left\{ \begin{array}{c} r\text{-subcubical} \\ \text{diagram of} \\ \text{punctured knots } E_S \end{array} \right\}}$$

There are maps

$T_r \longrightarrow T_{r-1}$ (containment of diagrams)

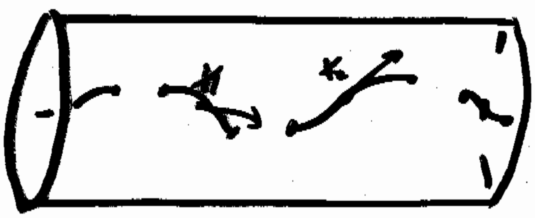
$\mathcal{K} \longrightarrow T_1$ (isotopies are constant.)

<u>Note</u>: $\mathcal{K}$ is the <u>limit</u> of the diagram.

A cosimplicial model for $\mathcal{K}$

E_s are configuration spaces:

$$
\begin{array}{ccc}
 & & \text{configurations with vectors attached} \\
 & & \| \\
E_s & \xrightarrow{\simeq} & FT(|s|-1,\ I \times D^{n-1})
\end{array}
$$



$$e \longmapsto \left(e(x_i),\ \frac{e'(x_i)}{|e'(x_i)|} \right)$$

$\Rightarrow$ $\mathcal{K}$ approximated by

$$\operatorname*{holim}_{s} FT(|s|-1,\ I \times D^{n-1})$$

Let $FT[k] =$ (a version of) Fulton-MacPherson compactification of $FT(k, I \times D^{n-1})$

Define

<u>Codegeneracies</u>: $s^i : FT[k] \longrightarrow FT[k-1]$

forget (x_i, v_i)

<u>Cofaces</u>: $d^i : FT[k] \longrightarrow FT[k+1]$

double (x_i, v_i) in direction v_i

<u>Thm (Sinha)</u>:

- $FT[\bullet] = \left\{ FT[0] \rightrightarrows FT[1] \underset{s^i}{\overset{d^i}{\rightrightarrows}} FT[2] \cdots \right\}$

is a cosimplicial space

- $\mathrm{Tot}^r FT[\bullet] \simeq T_r = \operatorname{holim}_S E_S$

$\left(\overset{\text{G-K-W}}{\Longrightarrow} \quad \mathcal{K} \simeq \mathrm{Tot}\, FT[\bullet] \right)$

Thm. (Lambrechts - V.):

FT[·] is *formal*, i.e. have

$$\begin{tikzcd}[column sep=small]
\cdots & (\Omega^* FT[k], d) \arrow[l, shift left] \arrow[l, shift right] & & (\Omega^* FT[k+1], d) \arrow[ll, shift left, "\Omega^*(d^i)"'] \arrow[ll, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right] \\
& & (\text{a simplicial DGA of graphs}) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & \\
\cdots & (H^* FT[k], 0) \arrow[l, shift left] \arrow[l, shift right] & & (H^* FT[k+1], 0) \arrow[ll, shift left, "H^*(d^i)"'] \arrow[ll, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

Also, $H^*(d^i)$ is easy to understand and write down.

Consequence:

$\exists$ second quadrant Bousfield-Kan H^* spectral sequence:

$$E_1^{-p,q} = H^q(FT[p])$$

$$d_1 = \Sigma(-1)^i \Omega^*(d^i) : E_1^{-p,q} \longrightarrow E_1^{-p+1,q}$$

$$(E_1, d_1) \underset{n>3}{\Longrightarrow} H^*(\mathrm{Tot}\, FT[\cdot]) \underset{n>3}{=} H^*(\mathcal{K})$$

Formality $\Rightarrow$ spectral sequence collapses at E_2

so $\quad H^*(\mathcal{K}) = H^*(E_1, d_1)$

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & \Omega^* FT[p+1] \arrow[u, "d"] \arrow[r, "\sum(-1)^i \Omega^*(d^i)"] & \Omega^* FT[p] \arrow[u, "d"] \arrow[r, "\sum(-1)^i \Omega^*(d^i)"] & \cdots\cdots \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}
$$

$p+1 \qquad\qquad p \qquad\qquad \cdots \quad 0$

$$\downarrow \cong$$

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & H^* FT[p+1] \arrow[u, "0"] \arrow[r, "\sum(-1)^i H^*(d^i)"] & H^* FT[p] \arrow[u, "0"] \arrow[r, "\sum(-1)^i H^*(d^i)"] & \cdots \\
 & \vdots \arrow[u, "0"] & \vdots \arrow[u, "0"] &
\end{tikzcd}
$$

$p+1 \qquad\qquad p$

Can also get a DGA model for $H^*(\mathcal{K};\mathbb{Q})$:

$$
\begin{tikzcd}
(\Omega^* \mathcal{K}, d) \\
(\Omega^* \operatorname{Tot} FT[\bullet], d) \arrow[u, "\simeq" right, "\text{G-K-W, Sinha}" {right, xshift=1.5em}] \arrow[d, "\simeq" right, "\text{Bott-Segal}" {right, xshift=1.5em}] \\
\operatorname{Tot} (\Omega^* FT[\bullet], d) \\
\cdot \arrow[u, "\simeq" right] \arrow[d, "\simeq" right, "\text{formality}" {right, xshift=1.5em}] \\
\operatorname{Tot} (H^* FT[\bullet], 0) \arrow[d, equal, "\text{def.}" right] \\
\left(\widetilde{\bigoplus_{p=0}} s^{-p} H^* FT[p], \; \sum (-1)^i H^*(d^i) \right)
\end{tikzcd}
$$

in category of simplicial DGA's

Can. simplify computations:

① Remove tangential data

$$\begin{tikzcd}
\overline{\mathcal{K}} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Tot}\, F[\bullet] \arrow[d] \\
\mathcal{K} = \mathrm{Emb}(I, I\times D^{n-1}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Tot}\, FT[\bullet] \arrow[d] \\
\Omega S^{n-1} = \mathrm{Imm}(I, I\times D^{n-1}) \arrow[r, "\simeq"] & \mathrm{Tot}\,(S^{n-1})^{\bullet}
\end{tikzcd}$$

($\overline{\mathcal{K}}$ looks like framed knots on π_0)

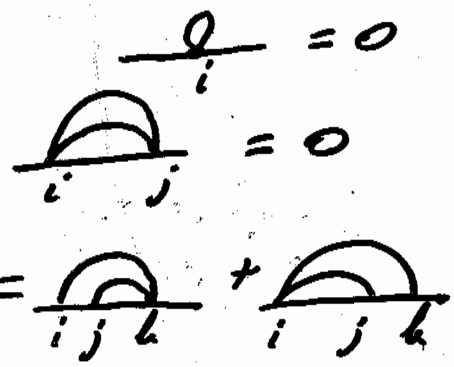
② Normalize $H^*(F[\bullet])$, i.e.

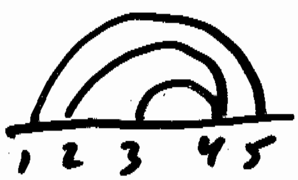
remove everything that's degenerate.

$$H^*(F[p]) = \mathbb{Q}[ij] \Big/ \begin{array}{l} \alpha_{ii} = 0 \\ \alpha_{ij}^2 = 0 \\ \alpha_{ij}\alpha_{jk} = \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij} \end{array}$$

$$1 \le i \le j \le p, \quad \deg \alpha_{ij} = n-1$$

$$\Longleftrightarrow \quad \mathbb{Q}[\text{chord diagrams}] \Big/$$



ex: 

$\longleftrightarrow \alpha_{15}\alpha_{24}\alpha_{34} \in H^{3(n-1)}F[5]$

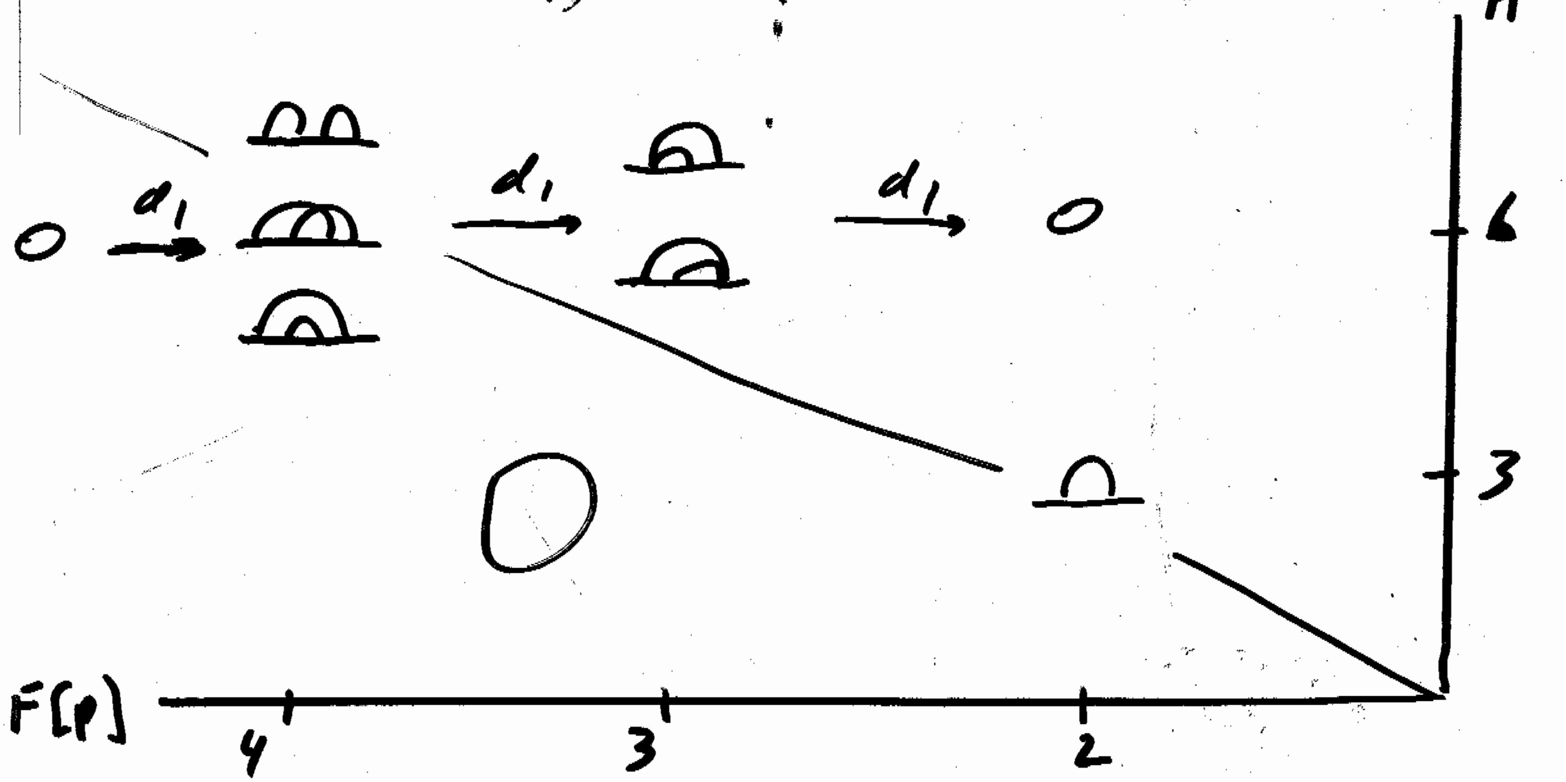
Normalization adds another relation
- all points must be used, e.g.

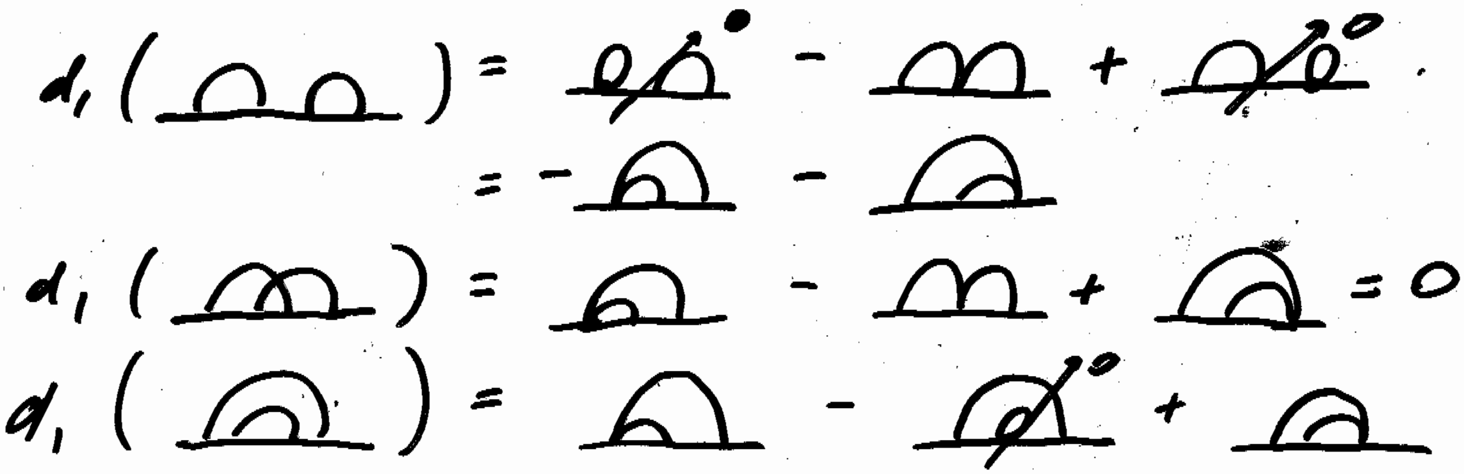
$0 =$ 

$\in H^{2(n-1)}F[4]$.

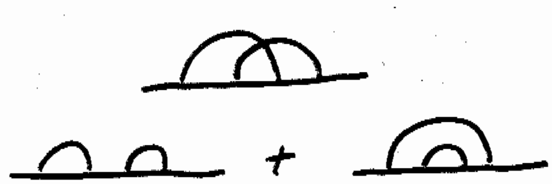
Using these relations, can easily write down normalized basis for any slot in E_1

ex: $H^2(\overline{K}_4) = \mathbb{Q}^2$





More generally,

deg	$n-3$	$2n-6$	$2n-5$
$H^*(\overline{K}_n)$		 +	

- In case of classical knots ($n=3$), still have the Goodwillie tower

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
\bar{K}_3 \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{Tot}^r F[\bullet] = \operatorname{holim}_{S \subseteq \{1,\dots,r\}} \left\{ \begin{array}{l} \text{diagram of} \\ \text{punctured} \\ \text{knots } E_S \end{array} \right\} \arrow[d] \\
 & \mathrm{Tot}^{r-1} F[\bullet] \arrow[d] \\
 & \vdots
\end{tikzcd}
$$

but do not have G-K-W convergence theorem.

Also, have H^* SS, but maybe not an isomorphism

$$H^* \mathrm{Tot}\, F[\bullet] \xleftarrow{\cong} H^* \mathrm{Tot}(\Omega^* F[\bullet])$$

(this is what the SS converges to)

<u>Thm (V)</u>:

$$H^0(\bar{K}_3) \supset V_r \xleftarrow{\cong} H^0 \mathrm{Tot}^{2r} \Omega^* F[\bullet],$$

where $V_r = \{$type r Vassiliev invariants$\}$

(also have $H^0 \mathrm{Tot}^{2r} \Omega^* F[\bullet] \cong H^0 \mathrm{Tot}^{2r+1} F[\bullet]$)