1/11/08 ①

Kontsevich's formality of little cubes operad and its consequences

University of Oregon Basic Notions Seminar

(In this talk, I'll present a theorem due to Kontsevich which is quite remarkable and which has far-ranging implications in various contexts. It'll take me a while to set it up and state, and then I'd like to spend some time on its proof. Unfortunately that may leave very little time for discussion of applications ...)

(This is a theorem about operads and rational homotopy theory, so let me start by talking about operads.)

An operad of spaces is a sequence of spaces $\{X_i\}_{i=1}^{\infty}$ with maps

$$X_n \times X_{j_1} \times X_{j_2} \times \cdots \times X_{j_n} \longrightarrow X_{\sum_{i=1}^{n} j_i}$$

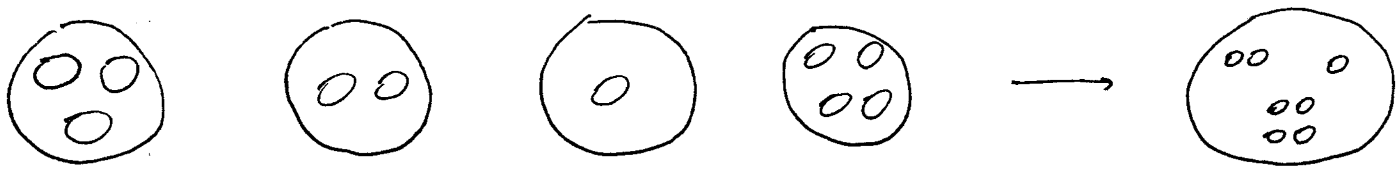
(These are very important in algebraic topology, and the first and most important example is:)

ex: <u>little balls (cubes) operad in $\mathbb{R}^d$</u>

X_i = space of i disjoint balls in unit ball of $\mathbb{R}^d$

(so a point in X_i is a configuration of i balls)

Maps are



(for our purposes, we need a different description of this operad:)

<u>Alternative description:</u> Let

$$C(n, \mathbb{R}^d) = \{ (x_1, x_2, \dots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \ \forall i,j \}$$

= <u>configuration space of n pts in $\mathbb{R}^d$</u>

$\left(\cong (\mathbb{R}^d)^n \setminus (\text{fat diagonal}) \right)$

(/ translation & scaling (optional))

Consider two maps

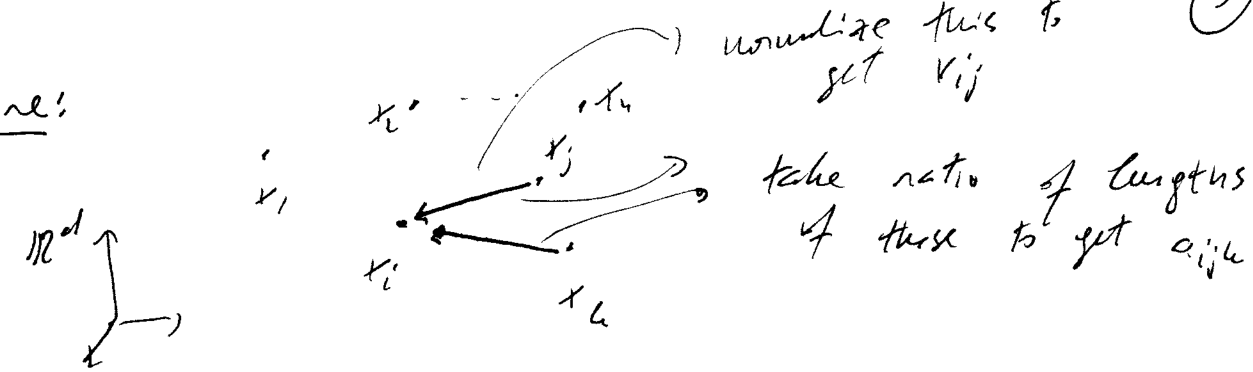
- $v_{ij} : C(n, \mathbb{R}^d) \longrightarrow S^{d-1}$, $1 \leq i < j \leq n$

  $(x_1, \dots, x_n) \longmapsto \dfrac{x_i - x_j}{|x_i - x_j|}$

- $a_{ijk} : C(n, \mathbb{R}^d) \longrightarrow [0, \infty]$, $1 \leq i < j < k \leq n$

  $(x_1, \dots, x_n) \longmapsto \dfrac{|x_i - x_j|}{|x_i - x_k|}$

picture:



Now have map

$$\delta : C(n, \mathbb{R}^d) \longrightarrow (S^{d-1})^{\binom{n}{2}} \times [0,\infty]^{\binom{n}{3}}$$

$$(x_1, \dots, x_n) \longmapsto \prod_{i<j} v_{ij} \times \prod_{i<j<k} a_{ijk}$$

Def (Kontsevich/Sinha):

Let $C[n, \mathbb{R}^d] = C[n]$ be the closure of the image of $(\delta \times Id)$ in $(S^{d-1})^{\binom{n}{2}} \times [0,\infty]^{\binom{n}{3}} \times C(n, \mathbb{R}^d)$

Properties:

Sinha
- $C[n] \simeq C(n)$ (good for doing topology)
- $C[n]$ is a (compact) manifold with corners (think cube; it has boundary of various dimensions)
- $C[n] \underbrace{\cong}_{\text{homeomorphism}}$ Fulton-MacPherson compactification of $C(n)$

(This space was considered, defined, and studied by Kontsevich, but he never really proved the above properties and was a little confused about $C[n]$ and a variant of this space where one takes away a_{ijk}; he thought this was the F-M compactification but wasn't. Sinha cleaned everything up carefully.)

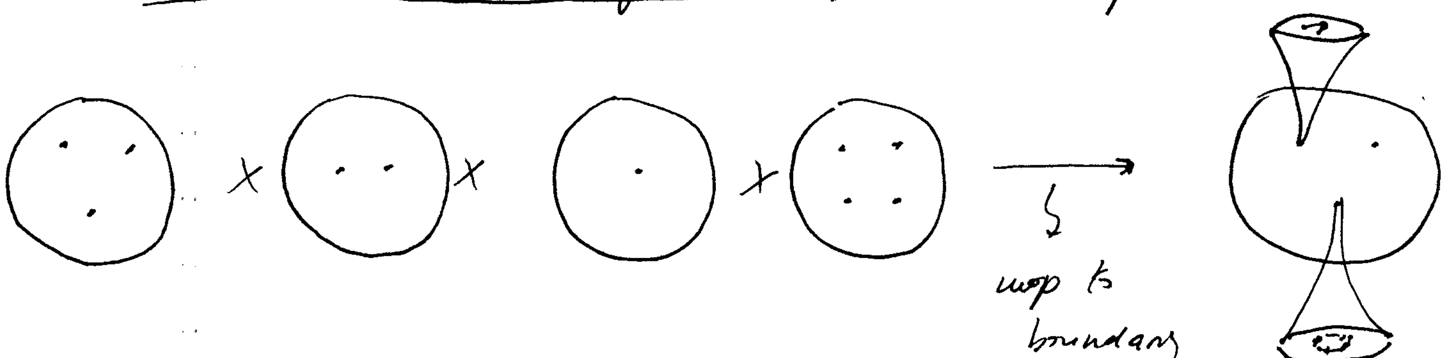
Boundary structure:

- points are allowed to collide, but directions of collisions and relative rates of approach are kept
- codimension of boundary is the number of stages of collision (so 3 pts colliding at the same time is codimension 1; 2 pts colliding and third joining later is codim 2, etc.) | track of

(• ~~boundary~~ boundary structure is given by a category of trees (Sinha))

Get operad $C[\bullet, \mathbb{R}^d] = \{C(n)\}_{n=1}^{\infty}$, called the

Fulton-MacPherson operad, with maps



by shrinking the three disks and letting points come together, then "pasting" them into points of first disc

(Point is: Think ~~of~~ of points in boundary as conf. pts that are infinitesimally close to each other, so that when you "zoom in infinitesimally", you see configurations that came together – this picture is also due to Sinha)

> **Prop:** $C[\bullet]$ is homotopy equivalent to little balls operad

(So from now on I'll just refer to F-M operad but you can of little balls operad instead)

Consider two complexes over $\mathbb{R}$ (everything is over $\mathbb{R}$)

$(C_*(C[n]), d)$ = singular chain complex of $C(n)$ with usual boundary d

$(H_*(C[n]), 0)$ = complex of homology groups of $C[n]$ with 0-differential

(You can do this for each n and in fact, C_* and H_* can be applied to the operad maps in F-M operad, so that we in fact get two new operads, but now in category of chain complexes:)

$$C_*(C[n]) \times C_*(C[j_1]) \times \cdots \times C_*(C[j_n])$$

$\Big\downarrow$ have this map since C_* is a monoidal functor

$$C_*(C[n] \times C[j_1] \times \cdots \times C[j_n])$$

$\Big\downarrow$ C_*(original operad map)

$$C_*(C[\textstyle\sum_{i=1}^{n} j_i])$$

Same for H_*

So get operads $C_*(C[\bullet])$ and $H_*(C[\bullet])$.

<u>Kontsevich's Formality Theorem</u>: The operad of chains on little balls operad is formal, i.e. $C_*(C[\bullet])$ and $H_*(C[\bullet])$ are rationally quasi-isomorphic, i.e. there are maps of complexes

$$\left(H_*(C[n]), 0\right) \underbrace{\longleftarrow - - - \longrightarrow}_{\substack{\text{bunch of maps} \\ \text{potentially, going in} \\ \text{either direction}}} \left(C_*(C[n]), d\right)$$

which respect (commute with) operad maps and which induce isomorphisms on homology.

(<u>Note</u>:
- Of course homologies are isomorphic, but to have maps inducing the isomorphism is <u>hard</u>.
- "Maps" means graded homomorphisms which commute with differentials
- This in particular says that chains on configuration spaces are formal.
- Sounds like this Theorem is no big deal, but <u>it is</u>!)

Proof: Work dually, i.e. show configuration spaces are formal, i.e. show there are quasi-isomorphisms of DGAs (differential graded algebras)

$$\boxed{(H^*(C[n]), 0) \longleftarrow \underbrace{\mathcal{D}_n}_{\text{to be defined}} \longrightarrow (C^*(C(n)), d)} \quad \circledast$$

(Then dualize and check operad structure is preserved, or check cooperad structure and then dualize - this is a little trickier. However, I don't want to talk about this part because

1. It's easier than proving ⊛; just formal manipulations
2. I don't have time.

So really what I want to do is tell you a bit about ⊛, but keep in mind that Kontsevich's ultimate goal was to prove formality of operad of chains of little balls.)

(Note: Dual statement ⊛ got upgraded to a statement about DGA's - a DGA is a graded algebra (so graded ring with multiplication) which is also a (co) chain complex with compatible differential, i.e.

- $d^2 = 0$
- $d(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d(b)$ (Leibniz)

<u>Why is this good?</u> ~~[illegible]~~ Since everything is over $\mathbb{Q}$, this statement really belongs in the realm of <u>rational homotopy theory</u>. The main result/technique in this subject is that to any space X (with nilpotent π_1) one can associate a DGA (A, d) which completely captures the <u>rational homotopy type</u> of X. This (A, d) is quasi-iso to $(C^*(X), d)$ and is called ~~the~~ a (Sullivan) <u>minimal model</u>, where A is of the form $\underbrace{\Lambda V}_{\text{exterior algebra}}$ → vector space. and the differential lands in $\Lambda^{\geq 2} V$, the subspace of words of length at least 2.

The relationship is this: There are functors (Quillen pair) ~~[illegible]~~

$$\mathrm{Top}_{\pi_1 \text{ nilpotent}} \underset{|\cdot|}{\overset{A_{PL} \simeq C^*}{\rightleftarrows}} \mathrm{DGA}$$

$$X \longmapsto C^*(X) \simeq A_{PL}(X) \simeq (\Lambda V, d)$$

rational space $= |(A, d)| \longleftarrow (A, d)$

realization of $(\Lambda V, d)$ gives a space, and taking A_{PL} of that space ~~[illegible]~~ ~~[illegible]~~ has $(\Lambda V, d)$ as its Sullivan minimal model.

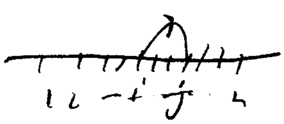
In any case, <u>everything starts with</u> $C^*(X) \simeq \underline{A_{PL}(X)}$ so this is what one needs to know about first, but <u>as DGA</u>; i.e. it is <u>important to understand $C^*(X)$ as a DGA, not just cochain complex.</u>)

(First, cohomology of configuration spaces:)

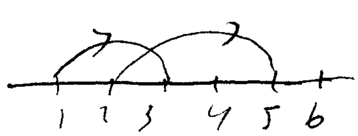
$$H^*(C[n], \mathbb{R}^d) \cong \underbrace{\Lambda}_{\text{exterior algebra}}(\alpha_{ij})/\sim, \quad 1 \le i \ne j \le n, \quad \deg \alpha_{ij} = d-1$$

(Recall that exterior algebra is algebra of words in α_{ij}'s s.t. $\alpha_{ij}^2 = 0$ and product is 0 whenever a generator α_{ij} shows up twice.)

Can represent each α_{ij} by a <u>chord diagram</u>



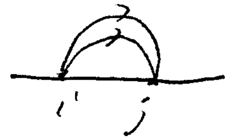
, so e.g.



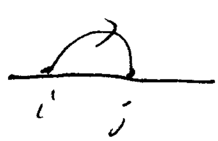
$\longleftrightarrow \alpha_{13}\alpha_{25} \in H^{2(d-1)}(C(6))$

(product is given by superimposing the chord diagrams; for correspondence always require that $\alpha_{ij}\alpha_{kl}$ has $i \le k$ and not $k < i$.)

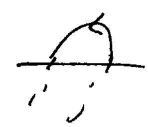
Relations $\sim$ then become

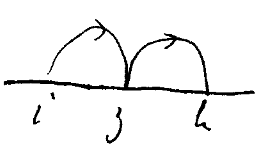


$= 0$



$= (-1)^d$





$\pm$

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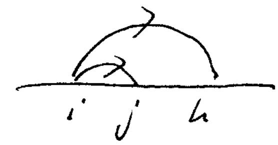


$= 0$

Arnold or 3-term relation

It turns out there is only one DGA in the middle of Kontsevich's Theorem, denoted by $\mathcal{D}_n$ – the algebra of admissible diagrams:

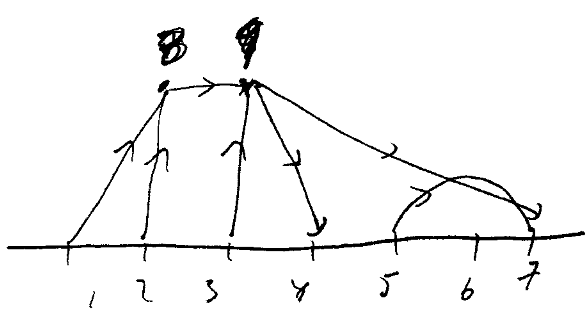
$$\boxed{(H^*(C[n]), 0) \xleftarrow{\ \mathcal{I}\ } \mathcal{D}_n \xrightarrow{\ I\ } (C^*(C[n]), d)}$$

Define $\mathcal{D}_n$:

An admissible diagram Γ on n external and q internal vertices is a graph with $n+q$ vertices, n of which are distinguished (external; pictured on a line segment) and some number of oriented, labeled edges connecting them, s.t.

- internal vertices have valence ≥ 3
- there are no loops
- every internal vertex is connected to an external one by a path.

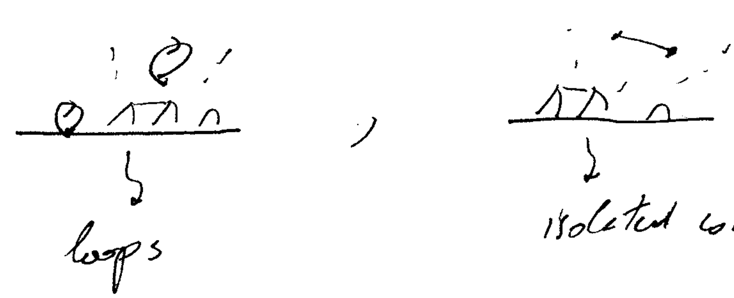
ex:



1–7 : external

8–9 : internal

(not allowed:



)

Let D_n = $\mathbb{R}$-span of admissible graphs with n external vertices modulo

- $\Gamma_1 = (-1)^d \Gamma_2$ if Γ_1, Γ_2 differ by orientation of edge (corresponds to $\alpha_{ij} = (-1)^d \alpha_{ji}$)

<u>Make D_n into a DGA by:</u>

① $\deg \Gamma = (d-1)n - dg$ (this gives grading)

② product is given by superimposing, i.e. aligning external vertices (gives algebra structure)

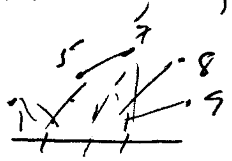
③ differential is given by contracting edges (and relabeling) with appropriate signs (gives cochain cplx structure)

<u>ex:</u>

$d\Big($  $\Big) =$  $\pm$  $\pm$ 

(corresponds to Arnold relation)

(relabeling is necessary when internal edge is contracted, e.g.

$d\Big($  $\Big) = \_\_ \pm \_\_ \pm$  $\pm \_\_$)

Not hard to show that:

- d increases degree by 1
- $d^2 = 0$

} so have cochain cplx

- d satisfies Leibniz rule $d(\Gamma_1 \Gamma_2) = d(\Gamma_1)\Gamma_2 \pm \Gamma_1 d(\Gamma_2)$ } so algebra structure is compatible w/ d

So get DGA $(\mathcal{D}_n, d)$

(Now I have to describe 2 maps of DGA's, I and $\bar{I}$:)

$$\boxed{\underline{\bar{I}\text{ (the easy one)}}:\ \left(H^*(C[n]), 0\right) \longleftarrow \mathcal{D}_n \ :\ \bar{I}}$$

(remember that cohomology on left is given by chord diagrams, so:)

chord diagrams $\longleftarrow\!\shortmid$ chord diagrams (no internal vertices)

$0 \longleftarrow\!\shortmid$ diagrams with internal vertices

$0 \longleftarrow\!\shortmid$ 

<u>Prop</u> (<u>not hard</u>): $\bar{I}$ is a DGA map.

(This means it respects the differential, which is fairly immediate by definition; the important thing is to send the Arnold relation to 0.)

$$\boxed{\underline{I\text{ (the hard one)}}:\ I: \mathcal{D}_n \longrightarrow \left(C^*(C[n]), d\right)}$$

This is defined in terms of <u>differential forms</u>.

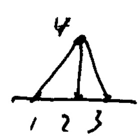
(So what I really mean by C^* is Ω^* (or A_{PL}), the cochain complex of (differential) forms on a space, i.e. the deRham complex. Taking homology of this complex gives <u>deRham cohomology</u>, which is really the version of cohomology that rational homotopy theorists work with most. The main feature of differential forms is that they can be integrated over spaces (or chains). Recall that the algebra of diff. n-forms in $\mathbb{R}^n$ is the exterior algebra on $dx_1, dx_2, \ldots, dx_n$ with exterior (wedge) product $dx_i \wedge dx_j$ which is anticommutative) with coefficients the 0-forms, or smooth functions on $\mathbb{R}^n$. The definitions extends easily to manifolds).

Given diagram $\Gamma \in D_n$ with $n+q$ vertices and m edges, have maps

$$\begin{tikzcd} C[n+q] \arrow[r, "\Phi_\Gamma"] \arrow[d, "\pi"', "\text{projection}"] & (S^{d-1})^m \\ C(n) & \end{tikzcd}$$

$$(x_1, \ldots, x_{n+q}) \longmapsto \prod_{\substack{\text{edges } ij \\ \text{of } \Gamma}} \frac{x_i - x_j}{|x_i - x_j|}$$

$$(x_1, \ldots, x_{n+q}) \longmapsto (x_1, \ldots, x_n)$$

(ex: For $\Gamma =$  , get

$$\Phi_\Delta : C[3+1] = C[4] \longrightarrow (S^{d-1})^3$$

$$(x_1, x_2, x_3, x_4) \longmapsto \prod_{j=2,3,4} \frac{x_1 - x_j}{|x_1 - x_j|}$$
)

Now:

- Let ω_{ij} be the unit volume $(d-1)$-form on the ijth sphere S^{d-1} (so a form which gives 1 when integrated over S^{d-1}).

- Pull back $\prod \omega_{ij}$ to $C[n+l]$ via Φ_Γ to get a form $\Phi_\Gamma^* \prod \omega_{ij}$.

- Push this form forward along π to $C[n]$, i.e. integrate $\Phi_\Gamma^* \prod \omega_{ij}$ along the fiber of π. This gives a form

$$\boxed{\alpha_\Gamma = \pi_* \Phi_\Gamma^* \prod \omega_{ij} \in \Omega^l(C[n]), \quad \text{where } l = \deg \Gamma \text{ (not hard to see)}}$$

$\Omega^l(C[n]) \cong C^*(C[n])$

(Integration along the fiber is defined as follows:

Value of form α_Γ on an l-chain σ in $C[n]$ is

$$\int_{\pi^{-1}\sigma} \Phi_\Gamma^* \prod \omega_{ij}. \quad)$$

(**Note**: If edge orientation is switched, get minus sign in ω_{ij}, so that's why there's a corresponding rel'n in D_n; this is the second time we needed this, but really for the same reason.)

Prop: I is a DbA map, i.e.

$$\textcircled{1} \quad d\,I(\Gamma) = I(d\Gamma)$$
$$\textcircled{2} \quad I(\Gamma_1 \Gamma_2) = I(\Gamma_1)\, I(\Gamma_2) \qquad (n \geq 2)$$

② is not hard (unravel the definitions)

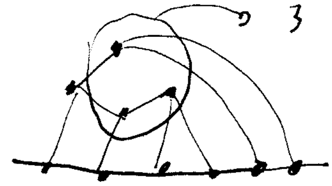
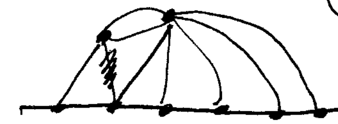
For ①: By Stokes' Thm,

$d\,I(\Gamma)$ = Integral along codimension 1 boundary of $C[n+g]$ of the restriction of $\Phi_\Gamma^* \prod \omega_{ij}$ to the boundary

(Now recall that codim 1 boundary of $C[n+g]$ is given by 2 or more points coming together at the same time. But for 2 points colliding, this precisely corresponds to the differential in D_n, i.e. contracting an edge in Γ and integrating gives precisely the form obtained by restricting the form corresponding to Γ to the boundary of $C[n+g]$ where the two points labeled by the contracted edge have collided. This means: Integral along boundary where 3 or more points collide must vanish – this is really the crux of the proof!

Vanishing arguments are ad hoc. Here's an example:)

3 points collide

Ex: $\Gamma =$  $\rightsquigarrow$  $= \Gamma'$

Then

$$\begin{tikzcd} \Phi_\Gamma\big|_{\Gamma'} : C[6+2] \arrow[rr, "\exists \text{ factorization}"] \arrow[dr] & & (S^{d-1})^8 \\ & (S^{d-1})^7 \arrow[ur] & \end{tikzcd}$$

since 2 maps are same due to double edge

So can pull back $8(d-1)$-form $\prod \omega_{ij}$ to $C[6+2]$ through $(S^{d-1})^7$, but this is $7(d-1)$-dim'l, so $\prod \omega_{ij}$ must pull back to 0.

Now have 2 DGA maps

$$H^*(C[n]) \xleftarrow{\bar{I}} D_n \xrightarrow{I} C^*(C[n])$$

Note that, for Γ a chord diagram, $\alpha_\Gamma = \prod \alpha_{ij}$ from before, which generate cohomology

which are surjective on generators, so to prove they're quasi-isos, it suffices to ~~show~~ prove

Prop: $H^*(D_n) \cong H^*(C[n])$

(Pf: Spectral sequence argument, not too hard)

$$\boxed{\text{This proves formality of configuration spaces!}}$$

To finish Kontsevich's Thm,

1. Put cooperad structure on $\{D_n\}$
2. Dualize to get formality of chains of configuration spaces
3. Check that duals of I and $\bar{I}$ commute with operad structure maps on C_*, H_*, and $(D_n)^*$.

(Remark: It would be nice to have an actual formality result for operads, i.e. a statement about cochains and cohomology rather than chains and homology. The problem is that C^* is not monoidal. What happens is that therefore a certain map ends up going the wrong way so one doesn't get cooperad maps. But we have

Thm (Lambrechts–V): $\exists$ commutative diagram

$$\begin{tikzcd}
D_n \arrow[rr, "I"] \arrow[dd, "\text{cooperad structure}"'] & & C^*(C[n]) \arrow[d, "\text{apply } C^* \text{ to operad maps in F-M operad}"] \\
 & & C^*(C[k] \times C[j_1] \times \cdots \times C[j_k]) \\
D_k \otimes D_{j_1} \otimes \cdots \otimes D_{j_k} \arrow[rr] & & C^*(C[k]) \otimes C^*(C[j_1]) \otimes \cdots \otimes C^*(C[j_k]) \arrow[u]
\end{tikzcd}$$

Künneth quasi-iso; goes the wrong way!!!)

Consequences

1. Kontsevich's formality in deformation quantization
2. Collapse of Vassiliev's spectral sequence for spaces of knots / combinatorial description of rational homology of spaces of knots (and homotopy)
3. Splitting of orthogonal calculus tower for spaces of embeddings of certain kind / rational invariance result for spaces of embeddings
4. Graph complex model for rational π_* of configuration spaces.
5. Development of PA rational homotopy theory

(1: Kontsevich proves

$$\boxed{\begin{array}{c}\text{Deligne Conjecture} \\ + \\ \text{formality}\end{array}} \Rightarrow \boxed{\begin{array}{c}\text{formality in} \\ \text{deformation} \\ \text{quantization}\end{array}}$$

I don't know anything about this except that it is physics-ispired.

Deligne Conjecture: A assoc. alg. / field k

Define Hochschild cplx $C^*(A,A)$ by

$$C^n(A,A) := \mathrm{Hom}(A^{\otimes n}, A) \quad , \quad n \geq 0$$

where

d is given by

$$d\phi(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^{n} (-1)^i \phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$
$$\pm a_1 \phi(a_2 \otimes \cdots \otimes a_{n+1})$$
$$\pm \phi(a_1 \otimes \cdots \otimes a_n) a_{n+1}$$

Deligne conjecture: $C_*(\mathcal{C}(\bullet, \mathbb{R}^2))$ acts on $C^*(A, A)$.

(~~Pf~~: Kontsevich, McClure-Smith, Voronov.)

((2): Let $\mathcal{K}_n = \mathrm{Emb}(S^1, \mathbb{R}^n)$, $n \geq 3$
$=$ space of embeddings, i.e. knots.

Thm (Sinha): There exists a cosimplicial space

$$\mathcal{K}^\bullet = \left(C[0] \rightrightarrows C[1] \;\substack{\to\\ \to\\ \to}\; C[2] \;\cdots \right)$$

whose totalization is $\simeq$ to $\mathcal{K}_n$.

There are two spectral sequences associated to $\mathcal{K}^\bullet$, the Bousfield-Kan cohomology and homotopy SS's. The homology one was studied by Vassiliev and was conjectured by him to collapse rationally.

Thm (Lambrechts-Turchin-V) Both spectral sequences collapse$/_{\mathbb{Q}}$ at E^2 for $n \geq 4$.

Cor: $H^*(\mathcal{K}_n) \cong$ Poisson operad in deg $n-1$ (which has easy combinatorial description)

This theorem follows from formality essentially because the vertical differential in the SS can be replaced by 0-differential:

$$\begin{array}{ccc} & \vdots & \\ & \uparrow d & \\ \cdots \xrightarrow{d} & \Omega^*(C[n]) & \xrightarrow{d} \cdots \\ & \uparrow d & \\ & \vdots & \end{array} \quad \xrightarrow{\simeq} \quad \begin{array}{ccc} & \vdots & \\ & \uparrow 0 & \\ \cdots \xrightarrow{d} & H^*(C[n]) & \xrightarrow{d} \cdots \\ & \uparrow 0 & \\ & \vdots & \end{array}$$

(③: One can generalize to $\mathrm{Emb}(M, \mathbb{R}^n)$ where M is a manifold and n is suitably large. One then uses embedding and orthogonal calculus of functors to study this space and conclude, using formality)

Thm (Arone–Lambrechts–V): Orthogonal tower for $H_*\mathrm{Emb}(M, V)$ splits rationally.

(($\Longrightarrow$ rational homology SS for $\mathrm{Emb}(M, \mathbb{R}^n)$ collapses at E^1)

Cor: If $f: M_1 \to M_2$ induces iso on rat'l homology, then $\exists$ rational iso

$$H_*(\mathrm{Emb}(M_1, \mathbb{R}^n)) \cong H_*(\mathrm{Emb}(M_2, \mathbb{R}^n))$$

(4: This is due to Lambrechts - Turchin.
From 2, we have

$$\mathrm{Tot}\, \mathcal{K}^\bullet \simeq \mathcal{K}_n \quad , \; n \geq 4,$$

and also
$$\mathrm{Tot}\, H^* \mathcal{K}^\bullet \cong H^* \mathcal{K}_n \quad \text{(simplicial ring)}$$
$$\mathrm{Tot}\, \pi_* \mathcal{K}^\bullet \otimes \mathbb{Q} \cong \pi_* \mathcal{K}_n \otimes \mathbb{Q}$$

Also have $\quad D_n \xrightarrow[\cong]{I} H^*(C[n])$

Since $H^*(C[n])$ make up the simplicial ring $H^*\mathcal{K}^\bullet$, it makes sense to collect all the D_n's into the corresponding simplicial ring $D_\bullet$. To do this, each D_n has to be enlarged a little without changing its homology (and then extend I ~~without~~ ~~changing~~ appropriately) in order for cofaces and codegeneracies to be defined appropriately.

But since $\mathcal{K}_n$ is an H-space (stacking long knots), the indecomposables in $H^* \mathcal{K}_n$ give the rational homotopy Lie algebra $\pi_* \mathcal{K}_n \otimes \mathbb{Q}$.

This is how graph complex giving homotopy of $\mathcal{K}_n$ is constructed.

Also taking indecomposables in $D_\bullet$ levelwise gives the graph cplx representation for $\pi_*(C[n]) \otimes \mathbb{Q}$.)

(5): Since spaces involved in the formality theorem are manifolds with corners, ~~[illegible]~~ ~~[illegible]~~ one cannot use standard deRham theory. Instead, one has to develop an appropriate corresponding theory for such spaces. In particular, one needs to know what integration along the fiber of

$$C[n+q] \longrightarrow C[n]$$ means.

It is for this reason that Kontsevich and Soibelman develop rational homotopy theory for semi-algebraic spaces such as $C[n]$. ~~[illegible]~~ Proofs and arguments are sketchy, and details will appear in a paper of myself and several coauthors. We will construct an algebra of PA forms analogous to A_{PL} and prove that it is equivalent to A_{PL} for ordinary spaces.)