CONFIGURATION SPACE INTEGRALS AND TAYLOR TOWERS FOR SPACES OF KNOTS

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Abstract. We describe Taylor towers for spaces of knots arising from Goodwillie-Weiss calculus of the embedding functor and extend the configuration space integrals of Bott and Taubes from spaces of knots to the stages of the towers. We show that certain combinations of integrals, indexed by trivalent diagrams, yield cohomology classes of the stages of the tower, just as they do for ordinary knots.

Contents

1. Introduction

In this paper we use configuration space integrals to establish a concrete connection between the study of knots and Goodwillie-Weiss calculus of the embedding functor [24, 14]. We do this by factoring the Bott-Taubes map, well-known to knot theorists, through a tower of spaces arising from this theory.

In more detail, fix a linear inclusion of $\mathbb{R}$ into $\mathbb{R}^m$. We study *long knots*, namely embeddings of $\mathbb{R}$ in $\mathbb{R}^m$ which agree with this linear map outside of a compact set. The space of such knots, with compact-open topology, is homotopy equivalent to the space of *based* knots in S^m. These can be thought of as maps of S^1 "anchored" at, say, the north pole, or, as we prefer, maps of the interval I to S^m which are embeddings except at the endpoints. The endpoints are mapped to the north pole with the same derivative. It is not hard to see that this space of based knots is a deformation retract of the space of based knots in the sphere which are prescribed in a neighborhood of the north pole. The latter, on the other hand, is clearly homotopy equivalent to the space of long

1991 *Mathematics Subject Classification.* Primary: 57Q45; Secondary: 81Q30, 57R40.

Key words and phrases. knots, spaces of knots, calculus of functors, configuration spaces, chord diagrams, finite type invariants.

knots. Let $\mathcal{K}_m$ be the space of long knots in $\mathbb{R}^m$ or S^m, $m \geq 3$. To simplify notation, we will often set $\mathcal{K} = \mathcal{K}_3$ when we wish to distinguish the case of classical knots from all others.

At the heart of our results are Bott-Taubes configuration space integrals [6] which are used for producing cohomology classes on $\mathcal{K}_m$. They were originally defined for ordinary knots, i.e. embeddings of S^1 in $\mathbb{R}^3$, but the modification to long knots is straightforward [8]. The idea is to start with a chord diagram with $2n$ vertices joined by chords, evaluate a knot on as many points, and then consider n maps to spheres given by normalized differences of pairs of those points. Which points are paired off is prescribed by the chord diagram. Pulling back the product of volume forms on the spheres via a product of these maps yields a form on the product of $\mathcal{K}_m$ with a suitably compactified configuration space of $2n$ points in $\mathbb{R}^m$. This form can then be pushed forward to $\mathcal{K}_m$. Various arguments involving Stokes' Theorem and the combinatorics of chord diagrams in the end guarantee that the result is a cohomology class. This was first done by Altschuler and Freidel [1] and D. Thurston [19] for $\mathcal{K}$, and then generalized by Cattaneo, Cotta-Ramusino, and Longoni [9] to $\mathcal{K}_m$. We will recall the main features of Bott-Taubes integration in §3. We will not provide all the details since they can be found in D. Thurston's work [19] or the survey paper [22].

The other ingredient we need is the Taylor tower for $\mathcal{K}_m$ arising from the calculus of the embedding functor. One considers spaces of "punctured knots," or embeddings of the interval with some number of subintervals removed. These spaces fit into cubical diagrams whose homotopy limits define stages of the tower, or "Taylor approximations" to $\mathcal{K}_m$. For $m > 3$, the tower converges (see Theorem 2.1 for the precise statement) so it represents a good substitute for $\mathcal{K}_m$. We review the construction of the tower in some detail in §2. Since embedding calculus is the less familiar half of the background we require, we do not assume the reader has had previous exposure to it.

In §4, we then turn our attention to extending the Bott-Taubes integrals to the tower and deduce our main result, stated more precisely as Theorem 4.5.

Theorem 1.1. *Bott-Taubes integrals factor through the stages of the Taylor tower for $\mathcal{K}_m$, $m \geq 3$.*

One importance of this theorem is that the stages of the Taylor tower lend themselves to a geometric analysis which complements the combinatorics and integration techniques of Bott and Taubes. In particular, one might ask if *all* cohomology classes of spaces of knots arise through Bott-Taubes integration and proceed to look for the answer in the Taylor tower. Something along these lines has been done for the case of classical knots $\mathcal{K}$ where some, but not all, of the constructions and results presented here hold as well. In particular, Bott-Taubes integration produces knot invariants and it was shown in [23] that the Taylor tower for $\mathcal{K}$ in fact classifies finite type (Vassiliev) invariants. Theorem 1.1 plays a crucial in establishing this result. The hope is that examining the Taylor tower more closely will shed new light on finite type invariants and the slightly mysterious appearance of integration techniques in knot theory. Some more details will be given at the end.

1.1. **Acknowledgements.** I am grateful to Tom Goodwillie for the guidance he provided and knowledge he shared with me over the years. I am also endebted to Riccardo Longoni for his help with the combinatorics of chord diagrams, as well as to Dev Sinha, Pascal Lambrechts, and especially Greg Arone for comments and suggestions.

2. Goodwillie-Weiss construction of the Taylor tower for $\mathcal{K}_m$

Let M and N be smooth manifolds of dimensions m and n, and let $Emb(M,N)$ denote the space of embeddings of M in N. Weiss [24] (see also [13]) develops a certain tower of spaces for studying $Emb(M,N)$. Its stages T_r are constructed from spaces of embeddings of some simple codimension 0 submanifolds of M in N. Each T_r comes with a canonical map from $Emb(M,N)$ and to T_{r-1}, and is in principle easier to understand than $Emb(M,N)$ itself. Goodwillie and Weiss [14], using work of Goodwillie and Klein [12], then prove the following

Theorem 2.1. *If $n-m>2$, the map $Emb(M,N) \longrightarrow T_r$ is $(r(n-m-2)+1-m)$-connected.*

Since the connectivity increases with r, the inverse limit of the tower is weakly equivalent to $Emb(M,N)$. Spaces T_r are examples of "polynomial," or "Taylor," approximations of $Emb(M,N)$ in the sense of Goodwillie calculus.

The general definition of the stages of the Taylor tower can be found in [24, Section 5]. However, in the case of $\mathcal{K}_m$, the definition readily simplifies to a concrete construction which produces an equivalent tower, even for classical knots (the edge of the dimensional assumption in the above theorem) [13, Section 5.1]. We thus focus in some detail on the construction of the Taylor tower for spaces of knots and start with some general definitions.

Definition 2.2. A *subcubical diagram* C_r is a functor from the category of nonempty subsets S of $\{1,\ldots,r\}$ with inclusions as morhisms to spaces, i.e. it is a diagram of 2^r-1 spaces X_S so that, for every containment $S \subset S\cup\{i\}$, there is a map $X_S \to X_{S\cup\{i\}}$ and every square

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X_{S\cup\{i\}} \arrow[d] \\ X_{S\cup\{j\}} \arrow[r] & X_{S\cup\{i,j\}} \end{tikzcd}$$

commutes.

Now let $x_1,\ldots,x_r$, be the barycentric coordinates of the standard $(r-1)$-simplex, which we denote by $\Delta^{r-1}_{\text{holim}}$. Denote by Δ^S_{holim} the face of $\Delta^{r-1}_{\text{holim}}$ given by $x_i = 0$ for all $i \notin S$. Thus if $T \subset S$, we have an inclusion $\Delta^T_{\text{holim}} \hookrightarrow \Delta^S_{\text{holim}}$ of a particular face of Δ^S_{holim}.

Definition 2.3. The *homotopy limit* of an r-subcubical diagram C_r, denoted by $holim(C_r)$, is a subspace of the space of smooth maps

$$\prod_{\emptyset\neq S\subseteq\{1,\ldots,r\}} Maps(\Delta^S_{\text{holim}}, X_S)$$

consisting of collections of smooth maps $\{\alpha_S\}$ such that, for every map $X_S \to X_{S\cup\{i\}}$ in the diagram, the square

$$\begin{tikzcd} \Delta^S_{\text{holim}} \arrow[r, "\alpha_S"] \arrow[d, hook] & X_S \arrow[d] \\ \Delta^{S\cup\{i\}}_{\text{holim}} \arrow[r, "\alpha_{S\cup\{i\}}"] & X_{S\cup\{i\}} \end{tikzcd}$$

commutes.

Remark 1. We will want to define certain forms on our homotopy limits in §4 so we consider only smooth maps in the above definition, thereby obtaining differentiable spaces. If we had instead considered spaces of all maps from simplices, we would have obtained homotopy equivalent spaces. More on homotopy limits of diagrams in model categories can be found in [7, 10].

Since C_r contains C_{r-1}, there are projections $holim(C_r) \to holim(C_{r-1})$ for all $r > 1$. Further, if $X_\emptyset$ fits C_r as its initial space, i.e. it maps to all other spaces in C_r and makes all the resulting squares commutative (and hence it maps to $holim(C_r)$), the diagram

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[dr] & holim(C_r) \arrow[d] \\ & holim(C_{r-1}) \end{tikzcd} \tag{1}$$

commutes.

We can now define the Taylor tower for the space of knots. For $r > 1$, let $\{A_i\}$, $1 \le i \le r$, be a collection of disjoint closed subintervals of $I \subset \mathbb{R}$, indexed cyclically. For each nonempty subset S of $\{1, \ldots, \text{r}\}$, define the space of maps

$$E_S = \mathrm{Emb}(I - \bigcup_{i \in S} A_i, S^m)$$

which are smooth embeddings other than at the endpoints of I. The endpoints are, as usual, mapped to the north pole in S^m with the same derivative.

The E_S can be thought of as spaces of "punctured knots," and are path-connected even for $m = 3$ since any punctured knot can be isotoped to the punctured unknot by "moving strands through the holes". If $T \subset S$, there is a restriction $E_T \to E_S$ which simply sends a punctured knot to the same knot with more punctures. These restrictions clearly commute. We can thus make the following

Definition 2.4. Denote by EC_r the subcubical diagram sending S to E_S for all nonempty seubsets S of $\{1, ..., r\}$ and sending inclusions to restrictions.

The homotopy limit of this diagram is the central object of study here so we give some details about what Definition 2.3 means in this case. Keeping in mind that a path in a space of embeddings is an isotopy, a point in $holim(EC_r)$ is a list of embeddings and families of isotopies:

- an embedding $e_i \in E_{\{i\}}$ for each i;
- an isotopy $\alpha_{ij} \colon \Delta^1_{\text{holim}} \to E_{\{i,j\}}$ for each $\{i, j\}$ such that
$$\alpha_{ij}(0) = e_i|_{E_{\{i,j\}}}, \quad \alpha_{ij}(1) = e_j|_{E_{\{i,j\}}};$$
- a 2-parameter isotopy $\alpha_{ijk} \colon \Delta^2_{\text{holim}} \to E_{\{i,j,k\}}$ for each $\{i, j, k\}$ whose restrictions to the faces of Δ^2_{holim} are
$$\alpha_{ij}|_{E_{\{i,j,k\}} \times \Delta^1_{\text{holim}}}, \quad \alpha_{jk}|_{E_{\{i,j,k\}} \times \Delta^1_{\text{holim}}}, \quad \alpha_{ik}|_{E_{\{i,j,k\}} \times \Delta^1_{\text{holim}}};$$
and in general,
- each $(|S|-1)$-parameter isotopy $\Delta^{|S|-1}_{\text{holim}} \to E_S$ is determined on the face of $\Delta^{|S|-1}_{\text{holim}}$ by the restriction of a $(|S|-2)$-parameter isotopy of a knot with $|S|-1$ punctures to the same isotopy of a knot with one more puncture.

Since we chose a definite indexing for the subintervals A_i of I, $i \in \{1, \ldots, r\}$, and thus for spaces of punctured knots E_S, $S \subseteq \{1, \ldots, r\}$, there are canonical maps

$$holim(EC_r) \longrightarrow holim(EC_{r-1}), \quad r > 2.$$

Also, $\mathcal{K}_m$ maps to each E_S again by restriction. Every square face in the cubical diagram obtained by adjoining $\mathcal{K}_m$ in the missing corner of EC_r commutes, so that we get commutative triangles as in (1).

Definition 2.5. For all $r > 0$, let $\mathcal{H}_r = holim(EC_{r+1})$ be the *rth stage* of the *Taylor tower for the space of knots*,

(2)
$$\begin{tikzcd}
& \vdots \arrow[d] \\
& \mathcal{H}_{r+1} \arrow[d] \\
\mathcal{K}_m \arrow[ur] \arrow[r] \arrow[dr] \arrow[dddr] & \mathcal{H}_r \arrow[d] \\
& \mathcal{H}_{r-1} \arrow[d] \\
& \vdots \arrow[d] \\
& \mathcal{H}_1.
\end{tikzcd}$$

The tower is shown here with the canonical maps from $\mathcal{K}_m$. Note that Theorem 2.1 implies convergence of this tower to $\mathcal{K}_m$ as long as $m > 3$.

Remark 2. Each point in EC_r determines a knot as long as $r > 2$. In fact, we only need to know what the elements of such a compatible collection are in $E_{\{1\}}, \ldots, E_{\{r\}}$ in order to recover a knot. Thus $\mathcal{K}_m$ actually completes the subcubical diagram of punctured knots as its limit for $r > 2$. We are therefore in some sense attempting to understand $\mathcal{K}_m$, a limit of a certain diagram, by instead studying its homotopy limit.

Spaces $\mathcal{H}_r$ are precisely what $\mathcal{K}_m$ will be replaced by in the Bott-Taubes construction of the next section.

3. Bott-Taubes configuration space integrals

3.1. **Trivalent diagrams.** Before we turn to configuration space integrals, we give a very brief introduction to a class of diagrams which turns out to best keep track of the combinatorics associated to those integrals. More details can be found in [3, 9, 15].

Definition 3.1. Let *trivalent diagram of degree n* be a connected graph consisting of an oriented interval, $2n$ vertices, and some number of chords between them. The vertices lying on the interval are called *interval* and are connected to the rest of the graph by exactly one chord. The vertices not on the interval are *free* and have exactly three chords emanating from them.

Depending on whether we are working in $\mathbb{R}^m$ for m even or odd, our configuration space integrals may change sign due to a permutation of the configuration points, a permutation in the product of certain maps to spheres, or due to a composition of one of those maps with the antipodal map. These sign changes correspond to a permutation of the vertices or chords of a trivalent diagram, or change in the orientation of a chord (see discussion following Theorem 3.8 for more details). As in §4.1 of [9], we thus distinguish two classes of diagrams as follows.

Label the vertices of a trivalent diagram by $1, ..., 2n$, orient its chords, and let TD_n^o be the set of all trivalent diagrams of degree n with these decorations. Define TD_n^e in the same way except also label the chords. Let STU_e be the relation from Figure 1. The decorations on the three diagrams in the picture should be compatible: Since the diagrams are the same outside the pictured portions, the vertex labels and orientations of chords and identical there. This leaves chord labels. In the only part where diagrams S, T, and U differ, the chords are labeled as in the figure, with $b' = b$ if $b < a$ and $b' = b - 1$ if $b > a$. Same for c'. We follow this pattern outside the pictured parts, and again note that now the chords for T and U are the same as those in S. Thus each chord for T and U is labeled as the corresponding chord in S unless its label is greater than a, in which case it is decreased by one.

Finally let STU_o be the same relation as STU_e except the factor of $(-1)^{a+j+v}$ is taken away, as are all the chord labels.

Definition 3.2. Let $\mathcal{D}_n^o$ and $\mathcal{D}_n^e$ be real vector spaces generated by TD_n^o and TD_n^e, modulo the STU_o and STU_e relations, respectively, with

- Diagrams containing a chord connecting two consecutive interval vertices, diagrams containing a double chord, and diagrams connecting a vertex to itself are all set to zero.
- For $D_1, D_2 \in \mathcal{D}_n^o$ which differ in the orientation of chords, set $D_1 = (-1)^s D_2$, where s is the number of chords with at least one free end vertex whose orientation is different.
- For $D_1, D_2 \in \mathcal{D}_n^e$ which differ in the orientation and labels of chords, set $D_1 = (-1)^s D_2$, where s is sum of the number of chords with at least one free end vertex whose orientation is different and the order of the permutation of the chords.

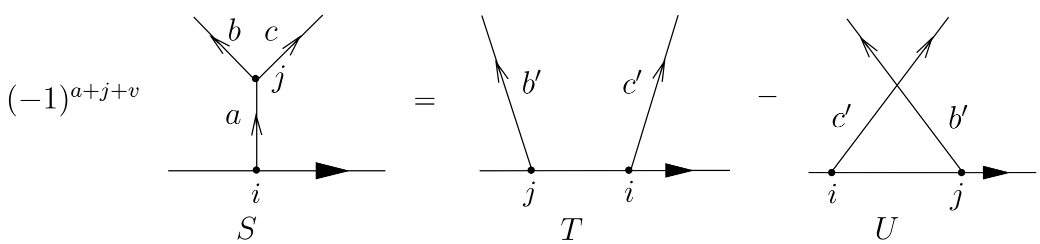

FIGURE 1. STU_e relation. The three diagrams agree outside the pictured portions. Here v is the number of interval vertices of the diagram S.

Remark 3. The relation which sets a diagram containing a chord connecting two consecutive interval vertices to zero is usually called the $1T$ (*one-term*) relation, and it is taken away if one considers *framed* knots. It is also intimately related to the correction term $M_D I(D_1, K)$ appearing in Theorem 3.8 and Theorem 4.5 [6, 19].

Let $\mathcal{D}^o = \oplus_{n>0} \mathcal{D}_n^o$ and $\mathcal{D}^e = \oplus_{n>0} \mathcal{D}_n^e$. Since our arguments do not depend of which space of diagrams is considered, we will just let $\mathcal{D}$ stand for either from now on and make some remarks on the parity where needed. Same for TD_n^o and TD_n^e which we will denote by TD_n.

Definition 3.3. Let $\mathcal{W}$ be the space of *weight systems* defined as the dual of $\mathcal{D}$. Let $\mathcal{W}_n$ be the degree n part of $\mathcal{W}$.

Theorem 3.4 ([3], Theorem 7)**.** *$\mathcal{D}$ and $\mathcal{W}$ are Hopf algebras.*

The product on $\mathcal{D}$ is given by continuing the interval of one diagram into another, and the coproduct is essentially given by breaking up the diagram into connected pieces (see Definition 3.7 of [3]). A consequence of the theorem is that it suffices to consider only *primitive* weight systems, as we will do from now on. These are precisely the weight systems which vanish on products of diagrams [3].

3.2. **Integrals and cohomology classes.** Recall that the linking number of two knots can be obtained by taking two points, one on each knot, and integrating over $S^1 \times S^1$ the pullback of the volume form on S^2 via the map giving the direction between those two points. Bott-Taubes configuration space integrals are in a way generalizations of this procedure to the case of a single knot. However, the points could now collide, so this configuration space has to be compactified for integration to make sense. Thus given a smooth manifold M of dimension m, let $F(k, M)$ be the configuration space of k distinct points in M and let $F[k, M]$ be its *Fulton-MacPherson compactification* [11, 2]. The standard way to define this space is through blowups of all the diagonals in M^k, but an alternative definition which does not use blowups was given by Sinha. We state it here in the relevant case of $M = \mathbb{R}^m$.

Definition 3.5 ([17], Definition 1.3)**.** Let $F[k, \mathbb{R}^m]$ be the closure of the image of $F(k, \mathbb{R}^m)$ in $(\mathbb{R}^m)^k \times (S^{m-1})^{\binom{k}{2}} \times [0, \infty]^{\binom{k}{3}}$ under the map which is the inclusion on the first factor and on the second and third sends the point $(x_1, ..., x_k)$ to the product of all $\frac{x_i - x_j}{|x_i - x_j|}$ and $\frac{|x_i - x_j|}{|x_i - x_l|}$, $1 \leq i < j < l \leq k$, respectively.

The compactification $F[k, \mathbb{R}^m]$ is a smooth manifold with corners of dimension km [17, §3], i.e. a space whose every point has a neighborhood homeomorphic to

$$\mathbb{R}^d \times \mathbb{R}_+^{km-d}$$

for some d and such that each transition function extends to an embedding of a neighborhood containing its domain. It is also compact in the more general case when M is compact. The configuration points in $F[k, \mathbb{R}^m]$ are allowed to come together while the directions as well as the relative rates of approach of the colliding points are kept track of. Codimension one faces (strata, screens), important for Stokes' Theorem arguments, are given by some number of points colliding at the same time. The combinatorics of these compactifications are very interesting and deep, and have been related to Stasheff associahedra and certain spaces of trees [18, §4].

To make Stokes' Theorem arguments work out, Bott and Taubes make the following definition.

Definition 3.6 ([6], page 5283)**.** Define $F[k, s; \mathcal{K}_m, S^m]$ as the pullback of

$$F[k, I] \times \mathcal{K}_m \xrightarrow{\text{evaluation}} F[k, S^m] \xleftarrow{\text{projection}} F[k+s, S^m].$$

These spaces are suitable for integration, as we have

Proposition 3.7 ([6], Proposition A.3)**.** *Spaces $F[k, s; \mathcal{K}_m, S^m]$ fiber over $\mathcal{K}_m$ and the fibers are smooth compact manifolds with corners.*

Each fiber of $F[k,s;\mathcal{K}_m,S^m]$ over $\mathcal{K}_m$ can be thought of as a configuration space of $k+s$ points in S^m with k of them constrained to lie on some knot $K\in\mathcal{K}_m$. The connection to trivalent diagrams is now clearer; the configuration points which can be anywhere in S^m can be represented by the free vertices while those which lie on a knot can be represented by the interval ones.

Since we wish to consider directions between points, we replace S^m by $\mathbb{R}^m\cup\infty$. This in turn replaces based knots in S^m by long knots in $\mathbb{R}^m$, but introduces "faces at infinity" discussed after Theorem 3.8 and in Lemma 3.9.

Now suppose a labeled trivalent diagram $D\in TD_n$ with k interval and s free vertices is given (so $k+s=2n$). A chord connecting vertices i and j gives a map

$$h_{ij}\colon F[k,s;\mathcal{K}_m,\mathbb{R}^m]\longrightarrow S^{m-1} \tag{3}$$
$$(p_1,...,p_i,...,p_j,...,p_{k+s})\longmapsto \frac{p_j-p_i}{|p_j-p_i|}.$$

The product of these maps over all $(k+3s)/2$ chords of D can be used for pulling back the product of unit volume forms ω_{ij}, call it ω, from the product of spheres S^{m-1} to $F[k,s;\mathcal{K}_m,\mathbb{R}^m]$. We denote the resulting $(k+3s)(m-1)/2$–form on $F[k,s;\mathcal{K}_m,\mathbb{R}^m]$ by α. Because of Proposition 3.7, it makes sense to push this form forward to $\mathcal{K}_m$, i.e. integrate it along the fiber of the map

$$\pi\colon F[k,s;\mathcal{K}_m,\mathbb{R}^m]\longrightarrow\mathcal{K}_m.$$

Finally let $I(D,K)$ stand for the pullback of ω followed by this pushforward $\pi_*\alpha$:

$$\begin{tikzcd}
\Omega F[k,s;\mathcal{K}_m,\mathbb{R}^m] \arrow[d, "\pi_*"] & \Omega(S^{m-1})^{(k+3s)/2} \arrow[l, "\prod_{\text{chords } ij} h^*_{ij}"'] \arrow[dl, "I(D,K)"] \\
\Omega\mathcal{K}_m &
\end{tikzcd}$$

Since the fiber of π has dimension $k+ms$, the resulting form on $\mathcal{K}_m$ has dimension

$$\frac{k+3s}{2}(m-1)-(k+ms)=(m-3)\frac{k+s}{2}=(m-3)n.$$

This is not necessarily a closed form. However, let D_1 be the diagram consisting of two interval vertices and one chord between them. We then have

Theorem 3.8. *For a nontrivial primitive weight system $W\in\mathcal{W}_n$, the map $T(W)\colon\mathcal{K}\to\mathbb{R}$ given by*

$$K\longmapsto\frac{1}{(2n)!}\sum_{D\in TD_n}W(D)(I(D,K)-M_DI(D_1,K)),$$

represents a nontrivial element of $H^{(m-3)n}(\mathcal{K}_m;\mathbb{R})$. Here M_D is a real number which depends only on D and $M_DI(D_1,K)$ vanishes for $m>3$.

In the case $m=3$, this theorem was first proved for ordinary closed knots by Thurston [19] and Altschuler and Friedel [1] who also show the zeroth cohomology class one gets this way on $\mathcal{K}$ is in fact a finite type n invariant. The generalization to $m>3$ is due to Cattaneo, Cotta-Ramusino, and Longoni, who also show the cohomology classes obtained this way are nontrivial [9, Section 6]. The proof does not depend on m except a little care has to be taken with signs.

Since a labeling of a diagram determines the labeling of configuration points in $F[k, s; \mathcal{K}_m, \mathbb{R}^m]$, changing the orientation of D may affect the signs of $I(D, K)$ and $M_D I(D_1, K)$ depending on m (orientation of the fiber might change). But the two diagram algebras $\mathcal{D}_n^e$ and $\mathcal{D}_n^o$, corresponding to m even and odd, are defined precisely so that W depends on the sign in the same way.

One proof of Theorem 3.8 is via Stokes' Theorem and proceeds by checking that the integrals on the boundary of the fiber of π either vanish or cancel out within the sum, so that the sum is in fact a closed form. Different arguments are used for various types of faces, which are called *principal* if exactly two points degenerate; *hidden* if more than two, but not all, points degenerate; and *faces at infinity* if one or more points approach infinity. The correction term $M_D I(D_1, K)$ comes from the possible contribution of the *anomalous* face corresponding to all configuration points coming together [22, Proposition 4.8]. While it is easy to see that the contribution is zero in case of knots in $\mathbb{R}^m$, $m > 3$ [22, Proposition 6.3], it is a conjecture that this is also the case for $m = 3$. D. Thurston [19] and Poirier [16] have computed it to be zero in some simple cases.

The vanishing arguments can be found in [6, 19, 16, 22] and can be written down very concretely using explicit coordinates on compactified configuration spaces [6, page 5286] (see also §4.1 in [22]). Integrals along principal faces do not necessarily vanish, but they can be grouped according to the STU relation (and another relation which follows from it, usually called the IHX relation; see [3, 22]). These integrals then cancel in the sum of Theorem 3.8 [22, §4.4]. For other faces, a key observation time and again is that the product of the maps h_{ij} factors through a space of lower dimension than the product of the spheres which is its initial target. Therefore α must be zero. This type of argument is illustrated in Lemma 3.9 below and it immediately takes care of the vanishing of integrals along hidden faces [22, Proposition 4.4] and faces at infinity where one or more of the points off the knot go to infinity [22, Proposition 4.7]. In case of long knots, however, there is an extra case of such a face corresponding to some points on the knot going to infinity. This cannot happen in the Bott-Taubes/Thurston setup since they consider closed knots. We deal with this case in the following lemma.

Lemma 3.9. *The pushforward $I(D, K)$ vanishes on the faces of the fiber of π corresponding to some or all points on the knot going to infinity.*

Proof. The argument is essentially that of Proposition 4.7 in [22]. Recall that our long knots are "flat" outside a compact set, i.e. they agree with a fixed linear inclusion of $\mathbb{R}$ in $\mathbb{R}^m$. Suppose a point p_i on the knot tend to infinity. If p_i is related to another point p_j by a map h_{ij} (meaning there is a chord connecting vertices i and j in D), then there are four cases to consider.

1. If p_j does not go to infinity, then h_{ij} restricts to a constant map along this face. The product of all such maps to $(S^{m-1})^{(k+3s)/2}$ then factors through $(S^{m-1})^{((k+3s)/2)-1}$. The pullback of ω to $F[k, s; \mathcal{K}_m, \mathbb{R}^m]$ thus has to be zero as does $I(D, K)$.
2. If p_j is on the knot and it also goes to infinity (regardless of whether it does so in the same direction as x_i), h_{ij} is constant on this face.
3. If p_j is off the knot and it also goes to infinity, but in a different direction than that of the fixed linear inclusion, h_{ij} is again constant.
4. If p_j is off the knot and it goes to infinity in the same direction and at the same rate as p_i, then p_j is either connected to a point p_k which does not, in which case h_{jk} restricts to a constant map on this face, or it does, in which case we look at all other points p_k is related to by maps. Since D is connected, there must eventually be two points for which the map

restricts to a constant map (if not, this means the entire configuration is translated along the knot to infinity and this is not a face).

□

We next modify the construction outlined in this section to the setting of the Taylor tower and generalize Theorem 3.8.

4. Generalization to the stages of the Taylor tower

Remember from §2 that a point h in $\mathcal{H}_k$ is a collection of families of embeddings parametrized by simplices of dimensions $0,\ldots,k$. The families are compatible in the sense that a k-simplex Δ^k_{holim} parametrizes a family of knots with $k+1$ punctures, while each of its faces parametrizes a family of knots with fewer punctures (how many and which punctures depends on which barycentric coordinates of Δ^k_{holim} are 0). However, the evaluation of a punctured knot on a point in $F[k,I]$ may not be defined since the configuration points may land in the parts of I that have been removed. To get around this, we will devote most of this section to the construction of a smooth map

$$F[k,I]\to\Delta^k_{\text{holim}}$$

whose graph will serve the purpose of choosing a punctured knot in the family $h\in\mathcal{H}_k$ depending on where the k points in I may be.

The interior of $F[k,I]$, the open configuration space $F(k,I)$, is given by points $(x_1,\ldots,x_k)$ which satisfy $0<x_1<x_2<\cdots<x_k<1$. Thus we have a natural identification $F(k,I)\simeq\Delta^k$, where Δ^k denotes the open k-simplex. Let Δ^k_{conf} be the closed simplex identified with the obvious compactification of $F(k,I)$, i.e. adding the faces to Δ^k. Also let $\partial_i\Delta^k_{\text{holim}}$ stand for the ith face of Δ^k_{holim} (ith barycentric coordinate is 0), and let $A(x)$ index the set of holes in which the configuration x may be. In other words,

$$A(x)=\{i : x_j\in A_i \text{ for some } \ j\}.$$

Proposition 4.1. *There is a smooth map $\gamma^k\colon\Delta^k_{conf}\longrightarrow\Delta^k_{holim}$, defined inductively, which depends on the choice of the punctures $A_1,...,A_r$ in I. Moreover, if $\gamma^i\colon\Delta^i_{conf}\to\Delta^k_{holim}$ has been defined for all $i<n$, then, for $1\le j\le n-1$, $\gamma^n\colon\Delta^n_{conf}\to\Delta^k_{holim}$ satisfies*

i)

$$\gamma^n(x_1,\ldots,x_{j-1},x_j,x_j,x_{j+2},\ldots,x_n)=\gamma^{n-1}(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_{n-1}),$$
$$\gamma^n(0,x_2,\ldots,x_n)=\gamma^{n-1}(x_2,\ldots,x_n)$$
$$\gamma^n(x_1,\ldots,x_{n-1},1)=\gamma^{n-1}(x_1,\ldots,x_{n-1});$$

ii) There exists an open neighborhood V of x and

$$\gamma^n(x')\in\bigcap_{i\in A(x)}\partial_i\Delta^k_{holim}\quad \textit{for all } x'\in V.$$

Conditions i) and ii) are required because of the following: Let $x=(x_1,\ldots,x_k)$, $0\le x_1\le\cdots\le x_k\le 1$ parametrize Δ^k_{conf}. The image in Δ^k_{conf} of two points coming together in $F[k,s;\mathcal{K}_m,\mathbb{R}^m]$ is $x_j=x_{j+1}$, $1\le j\le k-1$. This situation translates into the pushforward of a certain form along a principal face and we wish for integrals like this to cancel due to the STU and IHX relations after considering sums over all trivalent diagrams. The cancellation will only be possible if the integrals corresponding to each triple of diagrams have the same value when two points collide. However, one of the diagrams in the *STU* relation has fewer interval vertices, i.e. it is associated with the space $F[k-1, I]$. A way to ensure the appropriate integrals over $F[k, I]$ and $F[k-1, I]$ are equal is to define γ^k inductively based on the number of points in a configuration (keeping in mind that Δ^0_{conf} is a point) and to further impose condition i). The last two equations in i) are required for the integrals along the faces given by points colliding with the basepoint in S^m to cancel out.

As for condition ii), given t in Δ^k_{holim} and h in $\mathcal{H}_k$, one gets a point h_t in $\Delta^k_{\text{holim}} \times \mathcal{H}_k$ which is an embedding of the interval with up to k punctures. As mentioned at the beginning of this section, we want the evaluation of h_t on a configuration to be defined for all points $(q, t, h) \in \Gamma_k \times \mathcal{H}_k$. We therefore need that, whenever $x \in \Delta^n_{\text{conf}}$, $t = \gamma^n(x)$ is a point in Δ^k_{holim} such that the corresponding embedding h_t is defined for x. So depending on where x is in I, γ^n will map it to the interior or a face (or intersection of faces) of Δ^k_{holim} according to whether some of the x_j are in any of the removed subarcs A_i for $1 \le i \le k$. Condition ii) ensures this and more as it requires γ^n to map a *neighborhood* of every point x to the same face as x itself. This is needed for the resulting graph to be a smooth manifold with corners. Note that the intersection in condition ii) is nonempty since there is always at least one more hole in the interval than the number of points in a configuration (the number of configuration points is $n \le k$, while the number of holes in $\mathcal{H}_k$ is $k+1$).

Proof of Proposition 4.1. Assume smooth maps $\gamma^0, \ldots, \gamma^{n-1}$ have been defined on faces of Δ^n_{conf} and satisfy conditions i) and ii) (smoothness is needed for Stokes' Theorem). Then we can extend locally to a function γ^n on all of Δ^n_{conf}. However, we need to check that there are neighborhoods U_x for every point $x \in \Delta^n_{\text{conf}}$ so that the local extensions γ^n_x match on intersections.

Thus, given $x = (x_1, \ldots, x_n) \in \Delta^n_{\text{conf}}$ with all x_j distinct, pick a neighborhood U_x of x such that $A(x') \subset A(x)$ (no x' in U_x gets into holes x did not get into). This can be done since the A_i are closed subintervals. Then two intersecting neighborhoods are both mapped to the same face in Δ^k_{holim}, and so condition ii) is satisfied on intersections in this case. (Condition i) is vacuous here since we are in the interior of Δ^n_{conf}.)

If $x_j = x_{j+1}$, so that x is on a face $\Delta^{n-1}_{\text{conf}}$ of Δ^n_{conf}, choose a U_x so that its boundary in $\Delta^{n-1}_{\text{conf}}$ is contained in the neighborhood V from condition ii) for the point $x = (x_1, \ldots, x_j, \ldots, x_{n-1}) \in \Delta^{n-1}_{\text{conf}}$ and the map γ^{n-1}_x. Now γ^n_x, extended from V, maps the whole half-ball U_x to the same face in Δ^k_{holim}, and these match to define a function on intersections.

The preceeding easily generalizes to those x on lower-dimensional faces of Δ^n_{conf}. If there is more than one j for which $x_j = x_{j+1}$, choose U_x such that, *for each* j, the part of the boundary of U_x given by

$$U_x \cap \{x' : x'_j = x'_{j+1}\}$$

equals V, where V has been by induction determined by $x = (x_1, \ldots, x_j, \ldots, x_{k-1}) \in \Delta^{n-1}_{\text{conf}}$ and the map γ^{n-1}_x.

Thus γ^n can be defined locally. To define it as a smooth function on the whole n-simplex, let $\{U_\alpha\}$ be a finite open cover of Δ^n_{conf} given by neighborhoods U_x. Similarly,

$$\gamma^n_\alpha \colon U_\alpha \to \Delta^n_{\text{holim}}$$

are given by the maps γ_x^n. Let

$$\mu_\alpha : U_\alpha \to I,\ supp(\mu_\alpha) \subset U_\alpha,\ \sum_\alpha \mu_\alpha = 1,\ \mu_\alpha > 0,$$

be a partition of unity subordinate to the cover $\{U_\alpha\}$, and note that if two functions γ_α^n and γ_β^n both satisfy conditions i) and ii) on $U_\alpha \cap U_\beta$, so will their average, where averaging is done by the partition of unity. Thus setting

$$\gamma^m = \sum_\alpha \mu_\alpha \gamma_\alpha^n$$

produces a smooth map from the closed simplex Δ^n_{conf} to $\Delta^k_{\mathrm{holim}}$ satisfying i) and ii). $\square$

Remark 4. Instead of using $\mathcal{H}_k$ in this construction, $\mathcal{H}_j$ could have been used, for any $j > k$. Then $\Delta^j_{\mathrm{holim}}$ would parametrize a family of embeddings in $\mathcal{H}_j$, but we would only be interested in the subfamily parametrized by the face $\Delta^k_{\mathrm{holim}}$. There is no ambiguity as to which face is meant since the maps $\mathcal{H}_j \to \mathcal{H}_k$ are well-defined.

The space over which generalized Bott-Taubes integration will take place is now easy to define. Noting that there is a map

$$f\colon F[k, I] \longrightarrow \Delta^k_{\mathrm{conf}}$$

which is identity on the interior of $F[k, I]$ and forgets the extra information about the relative rates of approach of the colliding points, we have

Definition 4.2. Let

$$\Gamma_k = \{(x, t) : t = \gamma^k(f(x))\} \subset F[k, I] \times \Delta^k_{\mathrm{holim}}$$

be the graph of the composition

$$F[k, I] \xrightarrow{f} \Delta^k_{\mathrm{conf}} \xrightarrow{\gamma^k} \Delta^k_{\mathrm{holim}}.$$

Since $F[k, I]$ and $\Delta^k_{\mathrm{holim}}$ are manifolds with corners, it follows from our construction of γ^k that Γ_n is a manifold with corners for all $n \le k$. The generalization of the Bott-Taubes setup from the previous section is now straighforward. In analogy with Definition 3.6, we have

Definition 4.3. Define $\Gamma_{k,s}$ as the pullback

$$\begin{tikzcd}
\Gamma_{k,s} \arrow[r] \arrow[d] & F[k+s, \mathbb{R}^m] \arrow[d] \\
\Gamma_k \times \mathcal{H}_{k+s} \arrow[r] & F[k, \mathbb{R}^m].
\end{tikzcd}$$

Remark 5. Recall that for a point in the homotopy limit coming from a knot, all isotopies are constant. The manifold $\Gamma_{k,s}$ in this case is therefore precisely $F[k, s; \mathcal{K}_m, \mathbb{R}^m]$ from the Bott-Taubes setup (and an even more special case is $\Gamma_{k,0} = \Gamma_k = F[k, I]$).

Bott and Taubes' proof of Proposition 3.7, which they carry out in a very general setting, applies in our case, so that we immediately get an analogous statement

Proposition 4.4. *Spaces $\Gamma_{k,s}$ fiber over $\mathcal{H}_{k+s}$ and the fibers are smooth manifolds with corners.*

With Remark 4 in mind, we have chosen to construct $\Gamma_{k,s}$ as a bundle over $\mathcal{H}_{k+s}$ (see comment immediately following Theorem 4.5 for the reason why). We have also replaced S^m by $\mathbb{R}^m$ as before.

The fiber of the map $\Gamma_{k,s} \to \mathcal{H}_{k+s}$ can now thought of as follows: Recall that a point in $\mathcal{H}_{k+s}$ is parametrized by $\Delta^{k+s}_{\text{holim}}$. Given $h \in \mathcal{H}_{k+s}$ and depending on where the points of $F[k,I]$ are, a certain point $t \in \Delta^{k+s}_{\text{holim}}$ is chosen according to our construction. This gives a particular punctured knot h_t. A point in the fiber is then a configuration space of $k+s$ points in $\mathbb{R}^m$ with k of them constrained to lie on some punctured knot h_t. Note that the only difference between this and Bott-Taubes setup is a genuine knot $K \in \mathcal{K}_m$ is replaced by the punctured knots h_t.

Again given a trivalent diagram $D \in TD_n$ with k interval and s free vertices, there is a map

$$\Big(\prod_{\text{chords } ij} h_{ij}\Big) \colon \Gamma_{k,s} \longrightarrow (S^{m-1})^{(k+3s)/2} \tag{4}$$

given by normalized differences of those pairs of points in $\Gamma_{k,s}$ which correspond to pairs of vertices connected by chords in D. Each h_{ij} pulls back the volume form ω_{ij} to an $(m-1)$–form α_{ij} on $\Gamma_{k,s}$. The product of the α_{ij} is then a $(k+3s)(m-1)/2$–form α which can be pushed forward along the $(k+ms)$–dimensional fiber to produce an $(m-3)n$–form on $\mathcal{H}_{k+s}$. This time we denote the pullback followed by pushforward by $I(D,h)$. Let $M_D I(D_1,h)$ again be the correction term associated with the collision of all points in $\Gamma_{k,s}$, so that we may state a generalization of Theorem 3.8:

Theorem 4.5. *For a nontrivial primitive weight system $W \in \mathcal{W}_n$, $n > 1$, the map $T(W) \colon \mathcal{H}_{2n} \to \mathbb{R}$ given by*

$$h \longmapsto \frac{1}{(2n)!} \sum_{D \in TD_n} W(D)(I(D,h) - M_D I(D_1,h)) \tag{5}$$

represents a nontrival element of $H^{(m-3)n}(\mathcal{H}_{2n};\mathbb{R})$. The real number M_D again depends only on D and the correction term $M_D I(D_1,h)$ is zero for $m > 3$. If $h \in \mathcal{H}_{2n}$ comes from a knot, this is the usual Bott-Taubes map from Theorem 3.8.

Note that this is a restatement of Theorem 1.1. Also note that the degree $(m-3)n$ is in the range given by Theorem 2.1 for $m > 3$. It should now also be clear why $\Gamma_{k,s}$ was defined as a bundle over $\mathcal{H}_{k+s}$. Each trivalent diagram in the sum has a total of $k+s = 2n$ vertices. The domain of $T(W)$ should be the same space regardless of what k and s are. The proper space to define $T(W)$ on is thus $\mathcal{H}_{2n}$, since D can in the extreme case be a chord diagram with $s = 0$ and $k = 2n$.

The main point in Theorem 4.5 is that the Bott-Taubes map factors through the Taylor tower. Spaces $\Gamma_{k,s}$ have been constructed so that this is immediate (see Remark 5). To prove that the form on $\mathcal{H}_{2n}$ given by the map (5) is closed, one can repeat verbatim the arguments given in [6, 19, 22] proving that the form from Theorem 3.8 is closed. Since these arguments are lengthy but straightforward, we will not repeat them here. It suffices to say that the main reason why the arguments stay the same is that one can write down coordinates on $\Gamma_{k,s}$ in exactly the same way Bott and Taubes do on $F[k,s;\mathcal{K}_m,\mathbb{R}^m]$. These coordinates are for example given in equations (12) of of [22]. Everything in those equations stays the same except a knot K is replaced by a punctured knot h_t, as was already hinted at in the discussion following Proposition 4.4. But all the Stokes' Theorem arguments are based on these coordinates so that §4.2–4.6 in [22], where the vanishing results are proved, can now be repeated in exactly the same way. Since everything therefore immediately carries over from the setting of Theorem 3.8 to ours, it follows that the form given by (5) is closed.

To conclude, we briefly indicate how the extension of Bott-Taubes integration to the Taylor tower gives another point of view on finite type knot theory [3, 5]. The fact that configuration space integrals can be used to construct the universal finite type knot invariant has been known for some time [1, 19].

Note that Bott-Taubes integrals produce 0-dimensional cohomology classes, or knot invariants, in case of $\mathcal{K}$. Also recall that the Taylor stages $\mathcal{H}_k$ for $\mathcal{K}$ is defined the same way as for $\mathcal{K}_m$, $m > 3$. Let $\mathcal{H}_k^*$ be an algebraic analog for the Taylor stage, obtained by replacing the spaces of punctured knots by cochains on those spaces and taking the algebraic homotopy colimit of the resulting subcubical diagram (this colimit is the total complex of a certain double complex). One then has canonical maps

$$H^0(\mathcal{K}) \longleftarrow H^0(\mathcal{H}_{2n}) \longleftarrow H^0(\mathcal{H}_{2n}^*),$$

neither of which is necessarily an equivalence (the first because one no longer has Theorem 2.1). However, we have

Theorem 4.6 ([23], Theorem 6.10)**.** *$H^0(\mathcal{H}_{2n}^*)$ is isomorphic to the set of finite type n knot invariants.*

It is also not hard to see that one has isomorphisms between $H^0(\mathcal{H}_{2n}^*)$ and $H^0(\mathcal{H}_{2n+1}^*)$ [23, equation (34)] so that all the stages of the algebraic Taylor tower are accounted for. Thus its invariants are precisely the finite type invariants. Configuration space integrals and Theorem 4.5 are central to the proof of Theorem 4.6 but the isomorphism itself is given by a simple map based on evaluation of a knot on some points.

The Taylor tower is thus a potentially a rich source of information about finite type theory. One interesting question is whether the usual Taylor stages $\mathcal{H}_{2n}$ contain more than just the finite type invariants. This issue is very closely related to the conjecture that finite type invariants separate knots. Some further questions are posed in §6.5 of [23].

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