A SURVEY OF BOTT-TAUBES INTEGRATION

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Abstract. It is well known that certain combinations of configuration space integrals defined by Bott and Taubes [11] produce cohomology classes of spaces of knots. The literature surrounding this important fact, however, is somewhat incomplete and lacking in detail. The aim of this paper is to fill in the gaps as well as summarize the importance of these integrals.

Contents

1. Introduction

Let $\mathcal{K}_m$ be the space of smooth embeddings of S^1 in $\mathbb{R}^m$ and set $\mathcal{K}_3 = \mathcal{K}$ to simplify notation. There is a well-known way to produce cohomology classes on $\mathcal{K}_m$ coming from the the perturbative Chern-Simons theory. In case of classical knots $\mathcal{K}$, the construction yields special classes in zeroth cohomology, finite type knot invariants, which have been studied extensively over the past ten years by both topologists and physicists. The physics behind the construction will not be discussed here, but a good start would be the informative survey by Labastida [19].

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At the heart of the theory are configuration space integrals which first appeared some fifteen years ago in the work of Guadagnini, Martellini, and Mintchev [14] and Bar-Natan [4]. The version we are concerned with here is more recent and is due to Bott and Taubes [11]. The idea is that certain chord diagrams may be used as prescriptions for obtaining a form on a configuration space of points in $\mathbb{R}_m$, some of which are required to lie on a given knot, and then pushing this form forward to $\mathcal{K}_m$. Adding the result over all diagrams gives a closed form on $\mathcal{K}_m$. For nice overviews of the constructions in case of $\mathcal{K}$, see [7, 8, 22].

The most complete proofs of this fact, which has been used very effectively in recent years, are due to Altschuler and Freidel [1]. Among other things, they were the first to prove that this approach produces a universal finite type invariant (see §5). However, their arguments are inspired from physics and may not be transparent to an algebraic topologist. Same is true of the important work of Poirier [26, 27] who extended Bott-Taubes integrals to links and tangles (also see related work of Yang [36]).

On the other hand, the more topological arguments, which can be found in the original Bott-Taubes work as well as in D. Thurston's undergraduate thesis [31], are somewhat incomplete even if most of the necessary ideas are there. Further, Thurston's work was unfortunately never published. This paper owes much to Bott-Taubes and Thurston, and most of the arguments used here come directly from, or are inspired by, the ideas they give in [11] and [31].

The focus of Thurston's work is on $\mathcal{K}$, but the generalization to $\mathcal{K}_m$, originally due to Cattaneo, Cotta-Ramusino, and Longoni [12], is straightforward. Depending on m, however, one now obtains cohomology classes in different dimensions. The goal of this paper is to give detailed proofs of these facts. The statements we are after are Theorem 3.5, Theorem 5.3, and a weak version of Theorem 6.1, stated here loosely together as:

Theorem 1.1. *Bott-Taubes configuration space integrals combine to yield nontrivial cohomology classes of $\mathcal{K}_m$. For $\mathcal{K}$, they represent a universal finite type knot invariant.*

The bulk of the paper, §4, is devoted to proving the first part of this theorem. The proof is essentially Stokes' Theorem, where integration along various types of codimension one faces of the compactified configuration space has to be considered. The goal is to show that these integrals either vanish of cancel to give a closed form on $\mathcal{K}_m$. We do all this for $\mathcal{K}$ and then note in §6 that one only needs to introduce minor changes throughout for $\mathcal{K}_m$. The second part of Theorem 1.1 is addressed in §5, where we also review the basics of finite type theory. The material in §4 and §5 will hopefully fill the gap in literature mentioned above and provide a more topological treatment.

Along the way, we also introduce the algebra of trivalent diagrams in §2, motivate the definition of Bott-Taubes integrals in §3.1, and then take a necessary digression on Fulton-MacPherson compactification of configuration spaces in §3.2. The integration itself is described in §3.3.

Bott-Taubes configuration space integrals have played an important role in some recent developments in knot theory and beyond. For example, Cattaneo et. al. constructed a double complex from the set of chord diagrams and used this to show that, for any k, $\mathcal{K}_m$ has nontrivial cohomology in degrees greater than k for $m > 3$. Further, their double complex has has been related to the cohomology spectral sequence set up by Vassiliev [33] and converging to $\mathcal{K}_m$ [32]. This was used in [34] for showing the collapse of Vassiliev's spectral sequence along a certain line. In general, the hope is that Bott-Taubes integrals produce all cohomology classes of $\mathcal{K}_m$.

This spectral sequence result uses an extension of these integrals to "punctured knots" making up the stages of certain towers of spaces approximating $\mathcal{K}_m$ (in the sense of calculus of functors

[34]). This extension was also used to place finite type knot invariants in a more homotopy-theoretic framework [35]. We feel this work would benefit from a firmer grounding which we attempt to provide here.

Another active area of investigation is the relationship between Bott-Taubes integrals and the *Kontsevich Integral* [15], which is essentially the only other known way of producing a universal finite type invariant. Poirier characterized the relationship between the two approaches and reduced the question of their equivalence to the computation of a certain anomalous term (see discussion following Proposition 4.8). It is still an open and interesting question if this anomaly vanishes.

Analogous comparison occurs between various ways of producing invariants of rational homology 3-spheres and knots in rational homology 3-spheres. One approach is through a generalization of Bott-Taubes integration, as done by Bott and Cattaneo [9, 10], while the other ones are based on the Kontsevich Integral and come in many related variants [20, 5, 18]. The relationship between the two is still not well understood.

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2. Trivalent diagrams

Before we talk about configuration space integrals, we introduce a class of diagrams which turns out to nicely keep track of the combinatorics associated to those integrals. These diagrams (sometimes called Feynman) are frequent in Chern-Simons theory.

Definition 2.1. A *trivalent diagram of degree n* is a connected graph composed of an oriented interval and some number of line segments connecting $2n$ vertices of two types:

- *interval vertices*, constrained to lie on the interval, from which only one line segment emanates, and
- *free vertices*, or those not constrained to lie on the interval, from which exactly three line segments emanate.

If a line segment connects two interval vertices, it is called a *chord*; otherwise it is called an *edge*. The vertices are labeled $1, ..., 2n$, and each chord and edge is also oriented.

Let TD_n be the set of all trivalent diagrams of degree n. Let STU be the relation as in Figure 1.

Definition 2.2. Let $\mathcal{D}_n$ be the real vector space generated by TD_n, modulo the STU relation.

We next state a result, due to Bar-Natan [3] (see Theorem 6), which will be useful later and follows from the STU relation.

Proposition 2.3. *The identities given in Figures 2 and 3 also hold in $\mathcal{D}_n$. The second figure is meant to indicate that if D_1 and D_2 "close" by identification of the endpoints of the interval to the same circular diagram considered up to orientation-preserving diffeomorphism of the circle, then $D_1 = D_2$.*

Labeling a diagram and orienting its chords and edges in different ways produces a potentially large number of non-isomorphic diagrams. We reduce this number by requiring that the chords

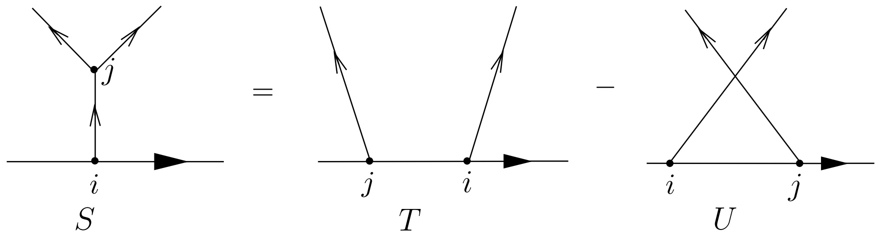


FIGURE 1. STU relation. The three diagrams here, as well as in the IHX relation below, are identical outside the pictured portions.

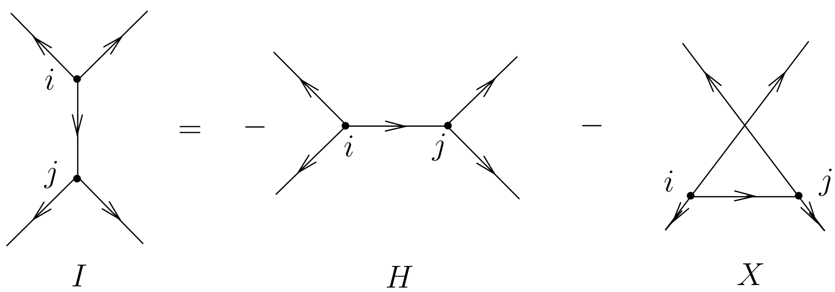


FIGURE 2. IHX relation

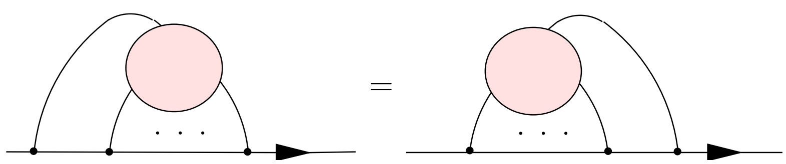


FIGURE 3. Closure relation

and edges are always oriented in such a way that they point from the vertex with lower label to the one with the higher label.

Let $\mathcal{D} = \oplus_{n>0}\mathcal{D}_n$. There is a pairing on $\mathcal{D}$, given by the operation of connected sum. Two diagrams are multiplied by continuing the interval of one into the interval of the other (with compatible orientation). Even though there are two ways to do this, the last relation in Proposition 2.3 ensures that the resulting diagrams are the same, so that the operation is commutative. The empty diagram serves as the identity. This operation is in correspondence with the operation of connected sum of knots.

Definition 2.4. A trivalent diagram which can be written as a product of two non-empty diagrams is *reducible*. Otherwise, the diagram is *prime*.

As it turns out, $\mathcal{D}$ also admits a coproduct, whose precise definition can be found in [3]. Bar-Natan also proves

Theorem 2.5 ([3], Theorem 7)**.** *$\mathcal{D}$ is a commutative and co-commutative Hopf algebra.*

Now define the space of *weight systems* $\mathcal{W}$ as the dual of $\mathcal{D}$ (and $\mathcal{W}_n$ as its *degree* n part). It also has the structure of a Hopf algebra. To understand $\mathcal{W}$, it therefore suffices to understand its primitive elements since they generate the whole algebra. To that end, we have the following useful statement, also due to Bar-Natan:

Proposition 2.6. *A weight system $W \in \mathcal{W}$ is primitive if and only if it vanishes on reducible diagrams.*

It thus suffices to consider weight systems only on prime diagrams since all others are obtained from these. This will be important in §4.3 and §4.4.

3. Bott-Taubes Integrals

Configuration spaces arise naturally in many embedding questions. The reason is that any embedding gives rise to a map of configuration spaces by evaluation of the embedding on some number of points. This is illustrated below by the simple example of the linking number of two classical knots. Bott-Taubes integrals can be thought of as generalizations of the Gauss integral which computes this integer invariant of 2-component links. We also use the next section to set the notation and make some basic definitions.

3.1. **The linking number.** Recall that, given a space X, the *configuration space* $F(k, X)$ is the subspace of X^k consisting of k distinct, ordered points in X:

$$F(k, X) = \{(x_1, \dots, x_k) \in X^k, \quad x_i \neq x_j \text{ for all } i \neq j\}.$$

Now let $F(1, 1; S^1, S^1)$ be the configuration space of one point on each of two disjoint circles, which is just the torus T^2, and let $K_1 \amalg K_2$ be an embedding of $S^1 \amalg S^1$ in $\mathbb{R}^3$. The evaluation map gives a composition

$$F(1, 1; S^1, S^1) \longrightarrow F(2, \mathbb{R}^3) \longrightarrow S^2 \tag{1}$$

$$(x_1, x_2) \longmapsto (K_1(x_1), K_2(x_2)) \longmapsto \frac{K_2(x_2) - K_1(x_1)}{|K_2(x_2) - K_1(x_1)|},$$

which we denote by by h_{12}.

Now let ω_{12} be the standard rotation-invariant unit volume 2-form on S^2,

$$\omega_{12} = \frac{x\, dydz - y\, dxdz + z\, dxdy}{4\pi(x^2 + y^2 + z^2)^{3/2}}. \tag{2}$$

The linking number is then defined as the integral of the pullback of ω_{12} by h_{12}:

$$L(K_1, K_2) = \int_{T^2} h_{12}^* \omega_{12}.$$

If the same procedure is applied to a knot, namely if we start with the configuration space of two points on a single circle, this $F(2, S^1)$ is an open cylinder so that the integration may not be defined. This can be remedied somewhat by adding the boundary to the cylinder. Geometrically, this corresponds to allowing x_1 and x_2 to "collide" (or "come together," or "degenerate," as we will interchangeably say). We can now use Stokes' Theorem to say that the Gauss integral will yield an invariant if the integral over the boundary of the cylinder of the restriction of $h_{12}^*\omega_{12}$ to that boundary vanishes. The restriction, which will be refered to throughout as *the tangential* *form*, is the pullback of ω_{12} via the extension of h_{12} to the boundary. We denote this extension by τ, which is readily seen to be given by

$$\tau = \frac{K'(x_1)}{|K'(x_1)|}. \tag{3}$$

However, the integral of $\tau^*\omega_{12}$ does not vanish so that the Gauss integral does not produce an invariant. One can now look for a "correction term" which would exactly cancel the contribution of the boundary integral. In this case, this term turns out to be given by the framing number of the knot [25].

This search for the "correction term" applies in the more general setup introduced by Bott and Taubes. Starting with a configuration space of any even number of points on the circle, they examine the various products of maps h_{ij} (analogous to h_{12} from above). To get nontrivial pullbacks, the configuration spaces first have to be compactified. However, in the case of more than two configuration points, the simplest compactification, which puts the fat diagonal back in, does not yield a boundary to which the pullbacks extend (the fat diagonal in X^k is the set $\{(x_1, ..., x_k) \in X^k : \exists i \neq j \text{ with } x_i = x_j\}$). What is needed, it turns out, is the more sophisticated compactification described in the next section. After this compactification is introduced, we will look at the Bott-Taubes method in more detail.

3.2. **Fulton-MacPherson compactification of configuration spaces.** The compactification of configuration spaces we use was first defined in the setting of algebraic geometry by Fulton and MacPherson [13]. It was then defined for manifolds by Axelrod and Singer [2] who used it in the context of Chern-Simons theory. We recall here the main features of the Axelrod-Singer version and use their constructions and definitions. Our notation, however, follows Sinha [29], whose alternative definition for the compactification does not use blowups. This makes his constructions similar to the one given by Kontsevich in [16] and perhaps more approachable for some purposes.

Let N be a smooth manifold. Another way to think of $F(k, N)$ is as an ordered product of k copies on N with all diagonals removed. The diagonals may be indexed by the sets $S \subseteq \{1, \ldots, k\}$ of cardinality at least 2. So let $\{x_1, x_2, \ldots, x_k\}$ be a point in $F(k, N)$ and let Δ_S denote the diagonal in N^k where $x_i = x_j$ for all $i, j \in S$. Finally let $Bl(N^k, \Delta_S)$ be the *blowup* of N^k along Δ_S, namely a replacement of Δ_S in N^k by its unit normal bundle. Since the interior of $Bl(N^k, \Delta_S)$ is $F(|S|, N)$, there are natural projections of $F(k, N)$ to $Bl(N^k, \Delta_S)$ for all S. Along with the inclusion of $F(k, N)$ in N^k, one then has an embedding of $F(k, N)$ in

$$N^k \times \prod_{\substack{S \subseteq \{1,\ldots,k\} \\ |S| \geq 2}} Bl(N^k, \Delta_S).$$

Definition 3.1. The *Fulton-MacPherson compactification* of $F(k, N)$, denoted by $F[k, N]$, is the closure of $F(k, N)$ in the above product.

When configuration points come together, the directions of approach are kept track of in $F[k, N]$. This is because the normal bundle records the tangent vectors of paths approaching a diagonal, but up to translation on the diagonal itself. In addition to translation, tangent vectors are also taken up to scaling because the normal bundle is replaced by its sphere bundle. This geometric point of view will be important in understanding the coordinates on $F[k, N]$ to be defined in §4.1.

We now list a few important properties of this compactification. Detailed proofs can be found in [2, 29]. For a more succinct treatment, see [30].

- The inclusion of $F(k,N)$ into $F[k,N]$ induces a homotopy equivalence.
- $F[k,N]$ is a smooth k-dimensional manifold with corners, and it is compact if N is compact.
- Any embedding of M in N induces an embedding of $F[k,M]$ in $F[k,N]$.

Since we will be integrating certain forms along the various codimension 1 faces of $F[k,N]$, it is worth discussing the stratification of this space in a little more detail. More explicit description can be found in [2].

Intuitively, a stratum (face, screen) corresponds to a subset A of the k configuration points degenerating. We denote this stratum by $\mathcal{S}_A$ and set $|A| = a$. To simplify notation, also assume the colliding points x_i are indexed by $1, 2, \ldots, a$. Either all x_i degenerate to the same point, or disjoint subsets of $(x_1, \ldots, x_a)$ degenerate to different points. Further, some points may degenerate faster than the others. The various ways of collisions of configuration points can be efficiently kept track of by *nested subsets* of $\{1, \ldots, a\}$. These are sets of subsets of $\{1, \ldots, a\}$, where each of the subsets contains at least two elements and each two subsets are either disjoint or one is contained in the other. Every stratum can be characterized in this way. So suppose that $\mathcal{S}_A$ is described by i nested subsets $\{A_1, A_2, \ldots, A_i\}$. Axelrod and Singer then show that the codimension of $\mathcal{S}_A$ in $F[k,N]$ is i. This is a direct consequence of an explicit description of coordinates on $F[k,N]$ (see §4.1). It follows that the codimension one strata consist of limits of sequences of points degenerating to the same point, at the same rate.

Remark. Given the representation of the strata as nested subsets, it is easy to see that $\mathcal{S}_A$ will be contained in $\mathcal{S}_{A'}$ as one of its faces if the first set of nested subsets is contained in the second. This can be used to graphically depict the relationships between the various strata of $F[k,N]$. If N is the interval, the pattern that emerges is that of Stasheff associahedra, which also appear in some combinatorial descriptions of finite type knot invariants. Alternatively, Sinha introduces a category of trees in [29] to study the stratification.

3.3. **Integrals and knot invariants.** To simplify the combinatorics to come, now let $\mathcal{K}$ be the space of maps of the unit interval I in $\mathbb{R}^3$ (or S^3; it will be clear form the context which we mean) which are embeddings except at endpoints and send the endpoints to the same point with the same derivative. This is clearly the same as the ordinary space of knots. To produce Gauss-like integrals that yield 0-forms, or knot invariants, Bott and Taubes first consider $F(4, S^1)$.

Given an embedding $K \in \mathcal{K}$, there are maps

$$h_{ij} \colon F[4,I] \times \mathcal{K} \longrightarrow S^2, \qquad i,j \in \{1,2,3,4\},\ i \neq j,$$

given by

$$(x_1, x_2, x_3, x_4) \longmapsto \frac{K(x_j) - K(x_i)}{|K(x_j) - K(x_i)|}.$$

Since $F[4,I]$ is four-dimensional, take a product of two of the maps h_{ij} so that the target, $S^2 \times S^2$ is also four-dimensional. Since Bott and Taubes do not consider framed knots, the only interesting choice (see §5 for an explanation) turns out to be $h = h_{13} \times h_{24}$. Let $\alpha = h^*(\omega_{13}\omega_{24})$ be the 4-form obtained by pulling back the product of unit volume forms ω_{13} and ω_{24} from $S^2 \times S^2$ via h. Probably the most important feature of the Fulton-MacPherson compactification for us is that α extends smoothly to the boundary of $F[k,I]$ for any k, as we will show in §4.1.

Let π be the projection of $F[4,I] \times \mathcal{K}$ to $\mathcal{K}$. Since this is a trivial bundle over $\mathcal{K}$ with 4-dimensional compact fiber $F[4,I]$, it makes sense to integrate α along this fiber. The composition

$$\begin{array}{ccc} \Omega(F[4,I] \times \mathcal{K}) & \xleftarrow{h_{13}^* \times h_{24}^*} & \Omega(S^2 \times S^2) \\ \downarrow{\scriptstyle \pi_*} & & \\ \Omega\mathcal{K} & & \end{array} \tag{4}$$

will thus produce a function on $\mathcal{K}$.

By Stokes' Theorem, the question of whether this function is a closed 0-form, i.e. a knot invariant, is now a question about the vanishing of the pushforward of α along the codimension one faces of $F[4,I]$:

$$d\pi_*\alpha = \pi_* d\alpha - (\partial\pi)_*\alpha, \tag{5}$$

where $(\partial\pi)_*\alpha$ means the sum of the pushforwards of α restricted to the codimension one faces. However, since each ω_{ij} is a closed form, so is α, and thus $\pi_* d\alpha = 0$. It follows that

$$d\pi_*\alpha = -(\partial\pi)_*\alpha. \tag{6}$$

Since $(\partial\pi)_*\alpha$ is not zero, however, one is led to search for a "correction term," namely another integral over a space with the same codimension one boundary as $F[4,I]$. A candidate presents itself immediately if we think of the first integral in terms of a chord diagram in Figure 4, where $x_1, \ldots, x_4$ represent the configuration points on the interval and the chords represent the way $h_{13} \times h_{24}$ pairs them off.

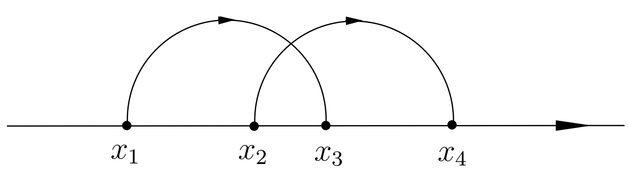


Figure 4. Diagram corresponding to integration along interior of $F[4,I]$

The stratum $x_2 = x_3$, for example, can then be pictured as another diagram as in Figure 5.

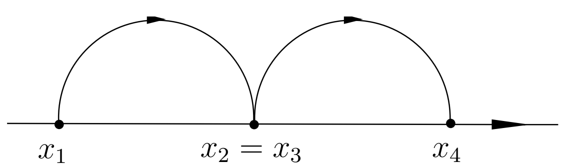


Figure 5. Diagram corresponding to integration along the face $x_2 = x_3$

Now consider Figure 6, the simplest diagram with a trivalent vertex. If we let $x_4 = x_2$ in that figure, we get exactly the same picture up to relabeling as the one depicting the stratum $x_2 = x_3$ from above. The space one needs to study then turns out to be a compactified configuration space of four points in $\mathbb{R}^3$, three of which are restricted to lie on the knot. Additionally, as the edges of the trivalent diagram suggests, there should be three maps to the sphere, h_{12}, h_{13}, and

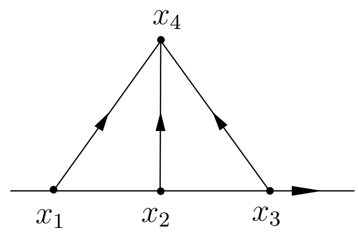


FIGURE 6. Simplest trivalent diagram

h_{14}, pulling back the volume forms. To construct the space itself, Bott and Taubes make the following general definition:

Definition 3.2. Define $F[k, s; \mathcal{K}, S^3]$ to be the pullback of

$$\begin{array}{ccc} & & F[k+s, S^3] \\ & & \downarrow \\ F[k, I] \times \mathcal{K} & \longrightarrow & F[k, S^3] \end{array}$$

where the vertical map is the projection onto the first k factors and the horizontal map is the evaluation of a knot in $\mathcal{K}$ on k configuration points in I.

The importance of these spaces is summarized in the following

Proposition 3.3 ([11], Proposition A.3)**.** *The space $F[k, s; \mathcal{K}, S^3]$ fibers over $\mathcal{K}$ and the fibers are compact manifolds with corners.*

A point in the fiber of $F[k, s; \mathcal{K}, S^3] \longrightarrow \mathcal{K}$ may then be thought of as a $(k + 3s)$-dimensional space of configurations of $k + s$ points in S^3, k of which are restricted to lie on a given knot $K \in \mathcal{K}$.

Since one cannot consider vectors between configuration points in S^m, one instead replaces S^m by $\mathbb{R}^m \cup \infty$, turning knots in S^m into knots in $\mathbb{R}^m$. This does not change any of the computations, except that one has to consider the *strata at infinity*, or the boundary components of $F[k, s; \mathcal{K}, \mathbb{R}^3]$ determined by configuration points tending to infinity.

Getting back to the trivalent diagram motivating Definition 3.2, the correction to the original integral $\pi_*\alpha$ turns out to be supplied by $F[3, 1; \mathcal{K}, \mathbb{R}^3]$ as predicted. More precisely, the edges in that diagram are thought of as giving a prescription for pulling back the volume forms ω_{14}, ω_{24}, and ω_{34} from three spheres and then pushing forward to $\mathcal{K}$ the resulting 6-form

$$\alpha' = \alpha_{14}\alpha_{24}\alpha_{34} = h_{14}^*\omega_{14}\, h_{24}^*\omega_{24}\, h_{34}^*\omega_{34}.$$

Bott and Taubes show that the pushforwards of α and α' along all strata, including those at infinity, either vanish or cancel out. We summarize in the following theorem (also proved by Bar-Natan using different methods in [4] and considered by Polyak and Viro in [28]):

Theorem 3.4 ([11], Theorem 1.3)**.** *The difference of pushforwards $\pi_*\alpha - \pi'_*\alpha'$ is a knot invariant.*

Thurston generalizes in [31] as follows: Suppose a labeled trivalent diagram D is given and suppose it contains k interval and s free vertices. The total number of its chords and edges is then $(k+3s)/2$. These can be used as prescriptions for setting up as many maps

$$h_{ij}\colon F[k,s;\mathcal{K},\mathbb{R}^3] \longrightarrow S^2.$$

The volume form may be pulled back to $F[k,s;\mathcal{K},\mathbb{R}^3]$ from each of the spheres, yielding a $(k+3s)$-form

$$\alpha = h_D^*\omega = \prod_{\substack{\text{chords and}\\ \text{edges } ij}} h_{ij}^*\omega_{ij}. \tag{7}$$

Here ω is the product of the volume forms and h_D the product of the h_{ij}. Integration along the $(k+3s)$-dimensional fiber of $F[k,s;\mathcal{K},\mathbb{R}^3]\to\mathcal{K}$ thus produces a function on $\mathcal{K}$, which we denote by $I(D,K)$. Let D_1 be the one-chord diagram and let $k+3s=2n$. We then have

Theorem 3.5 ([1, 31])**.** *For a primitive weight system $W\in\mathcal{W}_n$, $n\geq 1$, the map $T(W)\colon\mathcal{K}\to\mathbb{R}$ given by*

$$T(W)(K) = \frac{1}{(2n)!}\sum_{D\in TD_n} W(D)(I(D,K) - M_D I(D_1,K)),$$

where M_D is a real number which depends on D, is a knot invariant.

This is also a finite type n invariant, as was shown by Altschuler and Freidel [1] (see §5). The "correction" term $M_D I(D_1,K)$ will be discussed in detail in §4.6.

Remarks. Recall that chord diagrams are also considered to be trivalent, so that the sum in Theorem 3.5 includes those as well. Spaces $F[k,0;\mathcal{K},\mathbb{R}^3]$ simply reduce to $F[k,S^1]\times\mathcal{K}$ for those diagrams.

Next, fibers of $F[k,s;\mathcal{K},\mathbb{R}^3]\to\mathcal{K}$ inherit their orientation from $F[k+s,\mathbb{R}^3]$. Trivalent diagrams are also oriented by the labeling of their vertices (edges and chords, as required in §2, always point from the vertex with the lower label, and this means that the vectors h_{ij} must point in the corresponding directions). Since a labeling of a diagram determines the labeling of configuration points in $F[k,s;\mathcal{K},\mathbb{R}^3]$, changing the orientation of the diagram by permuting vertex labels may change the orientation of $F[k,s;\mathcal{K},\mathbb{R}^3]$. The corresponding integrals $I(D,K)$ and $I(D_1,K)$ will also differ in sign for two labeled diagrams with different orientations. But the weight system W also depends on the orientation, so that the signs will cancel out. Since the sum in the above theorem consists of $(2n)!$ identical expressions, the normalizing factor of $1/(2n)!$ is introduced.

We devote the next section to the proof of the above theorem.

4. Vanishing of Integrals Along Faces

We prove Theorem 3.5 by checking that the integrals $(\partial\pi)_*\alpha$ on the codimension one strata of $F[k,s;\mathcal{K},\mathbb{R}^3]$ either vanish or cancel out within the sum. Throughout this section, different arguments are used for various types of faces, which are, using the terminology of Bott and Taubes, called

- *principal*, if exactly two points degenerate,
- *hidden*, if more than two, but not all, points degenerate,
- *anomalous*, if all points degerate,

- *faces at infinity*, if one or more points approach infinity.

Using explicit coordinates on $F[k, s; \mathcal{K}, \mathbb{R}^3]$ we first show how α extends smoothly to such strata. Theorem 3.5 will follow as the combination of all the results in this section, at the end of §4.6. For the most part, the arguments used are given or suggested by Bott-Taubes and Thurston.

4.1. **Coordinates and pullbacks of forms on compactified configuration spaces.** We first describe coordinates on manifolds $F[k, 0; \mathcal{K}, \mathbb{R}^3] = F[k, I] \times \mathcal{K}$ and then on the $F[k, s; \mathcal{K}, \mathbb{R}^3]$. For the proof that we indeed get coordinates, see [2, 13]. The evaluation map

$$F[k, I] \times \mathcal{K} \longrightarrow F[k, \mathbb{R}^3]$$

is given on the interior of $F[k, I]$ by

$$(x_1, \ldots, x_k) \longmapsto p = (p_1, \ldots, p_k) = (K(x_1), \ldots, K(x_k)).$$

We now wish to extend this to the codimension one faces of $F[k, I]$. So let A be a subset of $\{1, \ldots, k\}$ containing a elements, and let q be a point in $F[k, I]$ where all x_i with $i \in A$ came together at the same time.

Then q is a configuration of the remaining $k - a + 1$ points as well as a number of unit vectors recording the (one of the two possible, for each pair) directions of approach of the a points. Applying K to q should thus yield a configuration of $k - a + 1$ points in $F[k, \mathbb{R}^3]$ as well as directions of approach of the colliding points $K(x_i)$. We denote the stratum in $F[k, \mathbb{R}^3]$ to which q belongs by $\mathcal{S}_{A,\emptyset}$.

Now parametrize the neighborhood of the a colliding points in $F[k, I]$ by

$$(x_1, u_1, u_2, \ldots, u_a, r\,;\, x_2, \ldots, x_{k-a+1}), \tag{8}$$

where

$$x_i \in I \text{ distinct}, \quad u_i \in \mathbb{R} \text{ distinct}, \quad r \geq 0, \quad \sum_{i=1}^{a} u_i = 0, \quad \sum_{i=1}^{a} |u_i|^2 = 1. \tag{9}$$

Then q is given in the neighborhood of the stratum in question by

$$\begin{aligned} q_i &= x_1 + r u_i, \quad i \in \{1, \ldots, a\}, \\ q_i &= x_{i-a+1}, \quad\ \ i \in \{a+1, \ldots, k\}. \end{aligned}$$

As r approaches 0, we are in the limit left with the configuration $(x_1, x_{a+1}, \ldots, x_k)$ as well as the direction vectors u_i.

Remarks. Notice that if two points on the interval, and later on a knot, collide, so will all points between them.

To simplify notation, we have chosen to label the colliding points by $1, \ldots, k$. However, we do not require that configuration points are ordered in any standard way, so that all the indices we use should be considered up to permutation.

This neighborhood in $F[k, I]$ maps to a neighborhood of $\mathcal{S}_{A,\emptyset}$ in $F[k, \mathbb{R}^3]$ consisting of points

$$\begin{aligned} p_i &= K(x_1 + r u_i), \quad i \in \{1, \ldots, a\}, \\ p_i &= K(x_{i-a+1}), \quad\ \ i \in \{a+1, \ldots, k\}. \end{aligned}$$

When $r = 0$, the remaining configuration is

$$(p_1, p_{a+1}, \ldots, p_k) = (K(x_1), K(x_2), \ldots, K(x_{i-a+1})),$$

while the unit vectors recording the directions of the collision are taken to be

$$\lim_{r\to 0}\frac{K(x_1+ru_j)-K(x_1+ru_i)}{|K(x_1+ru_j)-K(x_1+ru_i)|},\quad i,j\in\{1,\dots,a\},\quad i<j.$$

Upon expansion in Taylor series, this limit is easily seen to be

$$\tau=\frac{K'(x_1)}{|K'(x_1)|}. \tag{10}$$

In analogy with (3), we call τ the *tangential map.*

Finally notice that the conditions (9) also allow for the geometric interpretation of a point in $\mathcal{S}_{A,\emptyset}$ as an "infinitesimal polygon modulo translation and scaling."

To generalize to $F[k,s;\mathcal{K},\mathbb{R}^3]$, take the coordinates on its interior to be

$$(x_1,\dots,x_k,x_{k+1},\dots,x_{k+s}),$$

with

$$\begin{aligned} x_i\in I \text{ distinct},&\quad i\in\{1,\dots,k\}\\ x_i\in\mathbb{R}^3 \text{ distinct},&\quad i\in\{k+1,\dots,k+s\}.\end{aligned}$$

The points in the interior of $F[k,s;\mathcal{K},\mathbb{R}^3]$ will then be given by

$$(p_1,\dots,p_{k+s})=(K(x_1),\dots,K(x_k),x_{s+1},\dots,x_{k+s})$$

if we additionally require

$$K(x_i)\neq x_j,\quad 1\le i\le k<j\le k+s.$$

This condition prevents against a point "on the knot K" coinciding with a point "off the knot" (in $\mathbb{R}^3$). The terminology may be a bit misleading; keep in mind that a point "off the knot" is free to be anywhere in $\mathbb{R}^3$, including on the knot itself.

Now again assume for simplicity that the points on the knot are indexed by $1,\dots,k$, and those off the knot by $k+1,\dots,k+s$. Let A and B be subsets of $\{1,\dots,k\}$ and $\{k+1,\dots,k+s\}$ with cardinalities a and b. Denote by $\mathcal{S}_{A,B}$ the stratum of $F[k,s;\mathcal{K},\mathbb{R}^3]$ given by the coming together of the $a+b>1$ points indexed by the elements of A and B. Assume for now that A is nonempty, in which case the limit point is necessarily on the knot. A remark on the case $A=\emptyset$ follows (14).

For each A and B, we thus get a stratum $\mathcal{S}_{A,B}$. However, some of the strata are empty because if two points on the knot collide, so will all points between them. Thus if A contains indices i and j, but not all the indices of points between p_i and p_j, $\mathcal{S}_{A,B}$ is empty.

To give coordinates on a neighborhood of $\mathcal{S}_{A,B}$, assume as before that K is given and introduce parameters

$$(x_1,u_1,\dots,u_{a+b},r\,;\,x_2,\dots,x_{k+s-a-b+1}) \tag{11}$$

which satisfy the following conditions:

$$
(12)\qquad
\begin{aligned}
&(1) && x_i \in I \text{ distinct}, \quad 1 \le i \le k-a+1,\\
&(2) && x_i \in \mathbb{R}^3 \text{ distinct}, \quad k-a+2 \le i \le k+s-a-b+1,\\
&(3) && u_i \in \mathbb{R} \text{ distinct}, \quad 1 \le i \le a,\\
&(4) && u_i \in \mathbb{R}^3 \text{ distinct}, \quad a+1 \le i \le a+b,\\
&(5) && r \ge 0,\\
&(6) && K(x_i) \ne x_j, \quad i \le k-a+1, \quad j < k-a+1,\\
&(7) && K'(x_1)u_i \ne u_j, \quad i \le a, \quad j > a\\
&(8) && \sum_{i=1}^{a} |K'(x_1)|^2 u_i^2 + \sum_{i=a+1}^{a+b} |u_i|^2 = 1\\
&(9) && \sum_{i=1}^{a} u_i + \sum_{i=a+1}^{a+b} \frac{\langle K(x_1), u_i\rangle}{|K(x_1)|^2} = 0.
\end{aligned}
$$

Conditions (12.8) and (12.9) ensure that the limiting directions between colliding points are indeed vectors in the unit sphere bundle of the normal bundle, and can again be thought of as scaling and translation of vectors in $\mathbb{R}^3$. Condition (12.6) prevents against a point on the knot colliding with a point off the knot before the rest of the points join them, so that, along with (12.1) and (12.2), the stratum $\mathcal{S}_{A,B}$ is described by the $a+b$ points coming together exactly at the same time.

Configuration points near $\mathcal{S}_{A,B}$ $(r > 0)$ are then given by

$$
(13)\qquad
\begin{aligned}
&p_i = K(x_1 + ru_i), && i \in \{1,\dots,a\} && \text{(on knot, colliding)},\\
&p_i = K(x_{i-a+1}), && i \in \{a+1,\dots,k\} && \text{(on knot, not colliding)},\\
&p_i = K(x_1) + ru_{i-k+a}, && i \in \{k+1,\dots,k+b\} && \text{(off knot, colliding)},\\
&p_i = x_{i-a-b+1}, && i \in \{k+b+1,\dots,k+s\} && \text{(off knot, not colliding)}.
\end{aligned}
$$

Again remember that the indexing of points depends on the labeling of the configuration points as well as the sets A and B.

When $r = 0$, what is left is a configuration of $k+s-a-b+1$ points with the colliding points becoming $K(x_1)$. The limiting directions are

$$
(14)\qquad
\begin{aligned}
\tau = \frac{K'(x_1)}{|K'(x_1)|}, &\quad p_i, p_j \text{ on the knot},\\
\frac{u_j - K'(x_1)u_i}{|u_j - K'(x_1)u_i|}, &\quad p_i \text{ on the knot, } p_j \text{ off the knot},\\
\frac{u_j - u_i}{|u_j - u_i|}, &\quad p_i, p_j \text{ off the knot}.
\end{aligned}
$$

Constraint (12.7) prevents against the second vector being 0.

It should be clear how the above has to be modified in the case $A = \emptyset$. The colliding points are all off the knot know, so in particular the conditions (11) simplify. Some are vacuous, and condition (9) has to be changed to reflect the fact that x_1 should be a point in $\mathbb{R}^3$ off the knot "in the middle" of the colliding points. Same is true for the third equation in (13), while the first is not needed at all.

We are interested in how $\alpha = h_D^*\omega$ from (7) restricts to each $\mathcal{S}_{A,B}$. For this, it suffices to know how the maps h_{ij} behave on $\mathcal{S}_{A,B}$. The answer is now simple: If h_{ij} pairs two points, one or both of which are not among the colliding points, then it restricts to $\mathcal{S}_{A,B}$ as the normalized difference of two points in (13), with $r = 0$ (which two depends on whether the points are on or off the knot). If both points are among the colliding ones, h_{ij} extends to $\mathcal{S}_{A,B}$ as one of the maps in (14). Since these extensions are smooth, the pullback α also restricts smoothly to those codimension one strata given by collisions of points.

The remaining codimension one strata are given by one or more points going to infinity. We parametrize such strata the same way, except r now tends to infinity rather than 0. The limiting map between two points off the knot, p_i and p_j, going to infinity is then

$$h_{ij} = \lim_{r\to\infty} \frac{(x_1 + ru_j) - (x_1 + ru_i)}{|(x_1 + ru_j) - (x_1 + ru_i)|} = \frac{u_j - u_i}{|u_j - u_i|}, \tag{15}$$

which is exactly the same as the last map in (14), so nothing important changes in this case either. The map between a point p_j going to infinity and a point p_i which does not, also extends smoothly to the stratum at infinity as

$$h_{ij} = \lim_{r\to\infty} \frac{(x + ru_j) - p_i}{|(x + ru_j) - p_i|} = \frac{u_j}{|u_j|}. \tag{16}$$

In the following sections, we closely examine these maps to show that, for each A and B, the pushforward of α along $\mathcal{S}_{A,B}$ vanishes, cancels with others, or has to be compensated for by another integral.

4.2. **Faces determined by disconnected sets of vertices.** Given a labeled trivalent diagram with k interval and s free vertices, we now think of its vertex set as determing a space $F[k, s; \mathcal{K}, \mathbb{R}^3]$, and of its chords and edges as determining the map h_D. The strata $\mathcal{S}_{A,B}$ can also be thought of as being determined by subsets A and B of the vertex set since a labeling of the diagram determines a labeling of the configuration points in $F[k, s; \mathcal{K}, \mathbb{R}^3]$. Elements of A are interval vertices, while the vertices listed in B are free.

Suppose then a stratum is represented by a subset $A \cup B$ of a diagram's vetex set, and also suppose the chords and edges of the diagram are such that $A \cup B$ can be broken up into at least two smaller subsets such that no chord or edge connects a vertex of one subset to a vertex of another. We will say that such a set of vertices $A \cup B$ is *disconnected*, and denote the stratum corresponding to $A \cup B$ by $\mathcal{S}_{disc}$.

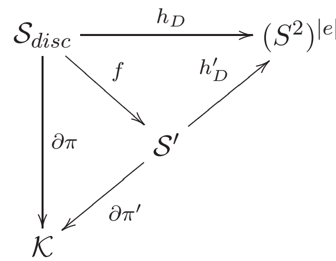
Proposition 4.1. *Unless $a = 2$ and $b = 0$, the pushforward of α to $\mathcal{K}$ along $\mathcal{S}_{disc}$ vanishes.*

The reason this statement is true is essentially that the two connected components can be translated independently. However, now write down the proof in some detail since we will refer to it often throughout. We first need the following lemma whose proof is left to the reader:

Lemma 4.2. *Suppose X and Y are spaces which fiber over Z with compact fibers of dimensions m and n, and suppose f is a fiber-preserving map from X to Y. Let β be a p-form on Y and let $\gamma = f^*\beta$. If $m > n$, then the pushforward of γ from X to Z vanishes.*

Proof of Proposition 4.1. Given a trivalent diagram D and a disconnected subset of vertices of D determining a stratum $\mathcal{S}_{disc}$, we will construct a space $\mathcal{S}'$ of dimension strictly lower then the dimension of $\mathcal{S}_{disc}$ such that the map h_D factors through $\mathcal{S}'$:

(17)



Suppose first that $\mathcal{S}_{disc}$ corresponds to a set of vertices $A \cup B$ with exactly two disconnected subsets. Let there be a_1 and a_2 interval vertices in each of those subsets, respectively, with $a_1 + a_2 = a = |A|$. Similarly, let $b_1 + b_2 = b = |B|$, where b_1 and b_2 are the numbers of free vertices in each subset. Assume without loss of generality that labeling of D is such that the a vertices are labeled by $1, \ldots, a$ and the rest by $a+1, \ldots, a+b$.

The parameters describing a neighborhood of the colliding points in $\mathcal{S}_{disc}$ are

$$(x, u_1, u_2, \ldots, u_{a+b}, r),$$

where $u_1, ..., u_a$ are in I, and $u_{a+1}, ..., u_{a+b}$ are in $\mathbb{R}^3$. For now, we omit the parameters for configuration points in $\mathcal{S}_{disc}$ which do not come together. We will get back to them later.

The u_i also must safisfy conditions (12) (with x_1 now replaced by x to simplify notation). Point x is in I unless $a = 0$, in which case x is a vector in $\mathbb{R}^3$.

Given $K \in \mathcal{K}$, the colliding points are

$$\begin{aligned} p_i &= K(x + ru_i), \quad i \in \{1, \ldots, a\} \\ p_i &= K(x) + ru_i, \quad i \in \{a+1, \ldots, a+b\}, \end{aligned}$$

as before. Recall also that the maps h_{ij} are given on $\mathcal{S}_{disc}$ by (14).

Now let $F[k, s; \mathcal{K}, \mathbb{R}^3]'$ be a space differing from $F[k, s; \mathcal{K}, \mathbb{R}^3]$ in that the blowups are not performed along all the diagonals involving points at least one of which corresponds to a vertex in one subset of $A \cup B$ and at least one of which corresponds to the other. Instead, the diagonals are simply put back in so that now not all directions of approach of points are recorded. This is still a compact manifold with corners since Fulton-MacPherson compactification can be constructed by a sequence of blowups along the diagonals and one obtains a manifold with corners at each stage [13, 2]. Thus $F[k, s; \mathcal{K}, \mathbb{R}^3]'$ is in fact a submanifold with corners of $F[k, s; \mathcal{K}, \mathbb{R}^3]$.

A part of the boundary of $F[k, s; \mathcal{K}, \mathbb{R}^3]'$ consists of the same $a + b$ points as in $\mathcal{S}_{disc}$ coming together. Denote this face by $\mathcal{S}'$ and consider its neighborhood described by two sets of independent parameters

$$(x_1, v_1, \ldots, v_{a_1+b_1}, r_1\,;\, x_2, , v_{a_1+b_1+1} \ldots, v_{a+b}, r_2). \tag{18}$$

If $a_1 > 0$, then x_1 is in I and it is in $\mathbb{R}^3$ if $a_1 = 0$. The same holds for a_2 and x_2. If both x_1 and x_2 are in I, we may assume $x_1 < x_2$. Each set of v_i must satisfy the conditions (12). Thus in particular, for the first set of parameters, we have

$$\sum_{i=1}^{a_1} |K'(x_1)|^2 v_i^2 + \sum_{i=a_1+1}^{a_1+b_1} |v_i|^2 = 1, \tag{19}$$

$$\sum_{i=1}^{a_1} v_i + \sum_{i=a_1+1}^{a_1+b_1} \frac{\langle K'(x_1), v_i\rangle}{|K'(x_1)|^2} = 0. \tag{20}$$

As usual, these parameters describe $a_1 + b_1$ points

$$p'_i = h_t(x_1 + r_1 v_i), \quad i \in \{1, ..., a_1\},$$
$$p'_i = h_t(x_1) + r_1 v_i, \quad i \in \{a_1 + 1, ..., a_1 + b_1\}.$$

The restrictions of the maps h'_{ij} to $\mathcal{S}'$ are identical to those for $\mathcal{S}_{disc}$ up to renaming of the parameters..

We now proceed to construct f for this subset of $A\cup B$. If x is a vector in $\mathbb{R}^3$, then x_1 will be as well, so set $x_1 = x$. The same happens if x is in I and the subset has an interval vertex. However, it may happen that x is in I (this means that $A \cup B$ contains at least one interval vertex), while x_1 is in $\mathbb{R}^3$ (this means that the subset of $A \cup B$ has no interval vertices). In this case, we set $x_1 = K(x)$. We also set $r_1 = r$.

Remember that we ultimately want

$$h'_{ij} \circ f = h_{ij} \tag{21}$$

when $r_1 = r = 0$. Note that h_{ij} can only be the tangential map (first map in (14)) when $A \cup B$ contains two interval vertices which are connected by a chord. But these two vertices then form a subset of $A \cup B$ which is disconnected from the rest of $A \cup B$. The first set of parameters in (18) can then be taken as (x_1, r_1), $x_1, r_1 \in I$. The only requirement f now has to satisfy is thus

$$\frac{K'(x_1)}{|K'(x_1)|} \circ f = \frac{K'(x)}{|K'(x)|},$$

and this is true since f sends x to x_1. An important consequence of this observation is Corollary 4.3.

We can thus assume the first subset of $A \cup B$ does not consist of two vertices with a chord between them, and we turn our attention to the remaining two maps in (14).

Suppose f gives $v_i = z_i$ for some z_i, $1 \le i \le a_1 + b_1$ (a_1 of these are numbers and b_1 are vectors). In order for (21) to hold for the last two maps in (14), the z_i should satisfy

$$z_j - K'(x)z_i = u_j - K'(x)u_i, \quad i \in \{1, \ldots, a_1\},\ j \in \{a_1 + 1, \ldots, a_1 + b_1\} \tag{22}$$

$$z_j - z_i = u_j - u_i, \quad i, j \in \{a_1 + 1, \ldots, a_1 + b_1\}. \tag{23}$$

Since the vertices in the subset of $A \cup B$ we are considering are connected by edges, it is easily seen that there will now be exactly one fewer independent equations then functions z_i. But these must also satisfy the constraints (19) and (20). The second constraint can be added to the system (22)–(23) which will now produce a unique solution. The z_i will be various combinations of vectors $u_j - u_i$, $u_j - K'(x)u_i$, $K'(x)$, and their magnitudes. They already satisfy the constraint (20), and to make sure they also satisfy (19), we may simply divide each z_i by

$$\left(\sum_{i=1}^{a_1} |K'(x_1)|^2 z_i^2 + \sum_{i=a_1+1}^{a_1+b_1} |z_i|^2 \right)^{\frac{1}{2}}.$$

The above expressions are never zero since

$$\begin{aligned} u_j \neq K'(x)u_i, \quad & i \in \{1, \dots, a_1\},\ j \in \{a_1+1, \dots, a_1+b_1\}, \\ u_j \neq u_i, \quad & i, j \in \{a_1+1, \dots, a_1+b_1\}. \end{aligned}$$

Modifying the z_i by these factors does not affect the compositions $h'_{ij} \circ f$ either. The factors cancel in h'_{ij} since they are positive, real-valued functions. The compositions are thus still h_{ij} as desired.

This procedure can be repeated to construct f for the other set of parameters in $\mathcal{S}'$, where f should set $x_2 = x$ or $x_2 = K(x)$, and $r_2 = r$.

For simplicity, the coordinates of the points in $\mathcal{S}_{disc}$ that do not come together have been omitted from the previous discussion. But the coordinates for those points are the same on $\mathcal{S}_{disc}$ and $\mathcal{S}'$, so f is the identity there, and it immediately follows that $h'_{ij} \circ f = h_{ij}$ for all the possible cases of one or both points outside of $\mathcal{S}_{disc}$.

We have thus constructed a space $\mathcal{S}'$ through which h_D factors. The parameters in (18), describing a neighborhood of $\mathcal{S}'$, determine, along with the constraints for each set of them, precisely as many dimensions as those describing a neighborhood of $\mathcal{S}_{disc}$ in $F[k, s; \mathcal{K}, \mathbb{R}^3]$. But f then restricts x_1, x_2, r_1, and r_2 and it follows that the fiber dimension of $\mathcal{S}'$ is smaller than that of $\mathcal{S}_{disc}$.

Lemma 4.2 thus completes the argument for this case of exactly two subsets of $A \cup B$ which have no chords or edges connecting them. The case of more subsets can be reduced to this situation by parametrizing the configuration points corresponding to all but one subset with one set of parameters. We are thus viewing some number of subsets as two, with one of them itself containing further subsets which are disconnected from each other. The only change in the argument is that there will now be fewer maps h_{ij} and h'_{ij} for this subset than before, and hence fewer conditions (22) and (23). The z_i will therefore not be unique. The number of maps h_{ij} is also the reason why this procedure does not work for a stratum determined by a single subset $A \cup B$ of the vertices of D which cannot be broken down into more subsets not joined by edges: If we try to reparametrize such a stratum by two sets of parameters, there will be more independent equations coming from the h_{ij} then variables z_i. □

It is easy to see why this proposition fails in the case of a principal face corresponding to a subset of the vertex set of D consisting of exactly two interval vertices which are not connected by a chord. The reason is essentially that, when two points in $\mathbb{R}^2$ come together at the blownup diagonal, the dimension of the resulting space is one, as it would have been had the blowing up not been performed, but rather the diagonal in $\mathbb{R}^2$ was put back in. This case, however, is taken care of in §4.4.

An easy consequence of Proposition 4.1 is as follows: Consider a diagram D with at least one chord. Then the vertex set of D is necessarily disconnected, with the two vertices joined by a chord forming a subset of the vertex set which is not connected to other vertices. Let $\mathcal{S}_D$ denote the anomalous face of $F[k, s; \mathcal{K}, \mathbb{R}^3]$, or the part of the boundary of $F[k, s; \mathcal{K}, \mathbb{R}^3]$ with all $k+s$ points coming together. We then immediately have

Corollary 4.3. *If $k + s > 2$ and a diagram D contains a chord, then the pushforward of $\alpha = (h_D)^*\omega$ to $\mathcal{K}$ along $\mathcal{S}_D$ vanishes.*

In particular, if D is a chord diagram (no free vertices), then the pushforward along $\mathcal{S}_D$ must be 0. (In fact, the same reasoning says that the pushforward along any hidden face for a chord diagram is zero, but this case is covered in the next section.) Corollary 4.3 will be used in Proposition 4.8.

4.3. **Hidden faces.** In this section, we prove

Proposition 4.4. *Suppose $\mathcal{S}_{A,B}$ is as before determined by the subset $A \cup B$ of the vertex set of a prime trivalent diagram D with $2n$ vertices. Suppose also $2 < |A \cup B| < 2n$. Then the pushforward of α to $\mathcal{K}$ along $\mathcal{S}_{A,B}$ vanishes.*

We first need two lemmas which are essentially due to Kontsevich [17].

Lemma 4.5. *Suppose $A \cup B$ contains vertices $i_1, \ldots, i_4$ with edges between them as in Figure 7. Each of the vertices i_3 and i_4 is either on the interval or free, and the two edges emanating upward from i_1 end in vertices not in $A \cup B$. Then the pushforward of α to $\mathcal{K}$ along $\mathcal{S}_{A,B}$ vanishes.*

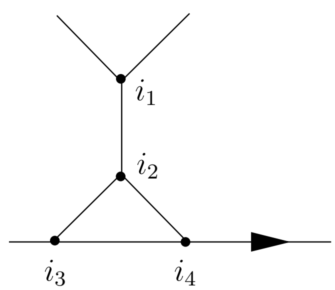


Figure 7. The case of Lemma 4.5

Proof. We will prove the statement by exhibiting an automorphism ϕ of $\mathcal{S}_{A,B}$ which preserves its orientation but takes α to $-\alpha$. The three maps corresponding to the pictured edges will be permuted, but they will all also be negated. Since the two pushforwards have to be the same, it will follow that $(\partial\pi)_*\alpha$ equals its negative and hence must be 0.

Recall that a neighborhood of $\mathcal{S}_{A,B}$ is parametrized by

$$(x_1, u_1, \ldots, u_a, u_{a+1}, \ldots, u_{a+b}, r\,;\, x_2, \ldots, x_{k+s-a-b+1}).$$

Assume first that both vertices i_3 and i_4 are free and consider a map ϕ from $\mathcal{S}_{A,B}$ to itself which leaves r, x_i alone and sends u_i to w_i given by

$$w_{i_1} = g_1(u_{i_3} + u_{i_4} - u_{i_1} + g_2 K'(x_1)), \tag{24}$$

$$w_{i_2} = g_1(u_{i_3} + u_{i_4} - u_{i_2} + g_2 K'(x_1)), \tag{25}$$

$$w_i = g_1(u_i + g_2), \qquad i \in \{1, \ldots, a\}, \tag{26}$$

$$w_i = g_1(u_i + g_2 K'(x_1)), \qquad i \in \{a+1, \ldots, a+b\} \setminus \{i_i, i_2\}, \tag{27}$$

and where

$$g_1 = \Bigg(\sum_{i=1}^{a} |K'(x_1)|^2 (u_i + g_2)^2 + \sum_{\substack{i=a+1 \\ i \neq i_1, i_2}}^{a+b} |u_i + g_2|^2 \tag{28}$$

$$+ |u_{i_3} + u_{i_4} - u_{i_1} + g_2 K'(x_1)|^2 + |u_{i_3} + u_{i_4} - u_{i_2} + g_2 K'(x_1)|^2 \Bigg)^{-\frac{1}{2}},$$

$$g_2 = -\frac{2\,\langle K'(x_1), u_{i_3} + u_{i_4} - u_{i_1} - u_{i_2} \rangle}{(a+b)\,|K'(x_1)|^2}. \tag{29}$$

To check that ϕ is indeed an automorphism, we need to show that it preserves conditions (12). All w_i, $i \in \{1, \ldots, a\}$, are distinct numbers as the corresponding u_i are. The same is true for vectors w_i, $i \in \{a+1, \ldots, a+b\} \setminus \{i_i, i_2\}$. Also, w_{i_1} is different from w_{i_2}, but one or both of them might equal some other vectors.

So suppose, for instance, that w_{i_1} equals w_i for some i. In this case, we simply go back to the construction of $F[k, s; \mathcal{K}, \mathbb{R}^3]$ and modify it so that the blowup along the diagonal $p_{i_i} = p_i$ is not performed. The effect on the parametrization of $\mathcal{S}_{A,B}$ is that the vectors u_{i_1} and u_i are no longer required to be distinct. Now the two points are allowed to come together before the others in $\mathcal{S}_{A,B}$ but, since there was no blowup along their diagonal, $\mathcal{S}_{A,B}$ still has codimension one in $F[k, s; \mathcal{K}, \mathbb{R}^3]$. The important observation is that this has no effect on the maps h_{ij}—there is no edge connecting i_i and i in D and hence no map relating the corresponding points, p_{i_i} and p_i. (This kind of a modification in the construction of $F[k, s; \mathcal{K}, \mathbb{R}^3]$ was also required in the proof of Proposition 4.1.)

By omitting some of the blowups, we can similarly preserve the condition

$$K'(x_1) u_i \neq u_j, \quad i \in \{1, \ldots, a\},\ j \in \{a+1, \ldots, a+b\}.$$

Now we do not impose this condition for $j = i_1$ and $j = i_2$ while, for the remaining j, it is still satisfied after ϕ is applied.

The condition

$$K(x_i) \neq x_j, \quad i \in \{1, \ldots, k-a+1\},\ j \in \{k-a+2, \ldots, k+s-a-b+1\},$$

trivially remains satisfied since ϕ is the identity on those parameters.

As for the remaining two conditions, constraint (12.8) maps under ϕ to

$$\sum_{i=1}^{a} |K'(x_1)|^2 (g_1(u_i + g_2))^2 + \sum_{\substack{i=a+1 \\ i \neq i_1, i_2}}^{a+b} |g_1(u_i + g_2)|^2$$

$$+ |g_1(u_{i_3} + u_{i_4} - u_{i_1} + g_2 K'(x_1))|^2 + |g_1(u_{i_3} + u_{i_4} - u_{i_2} + g_2 K'(x_1))|^2 = 1$$

Constraint (12.9) becomes

$$g_1 \Bigg(\sum_{i=1}^{a} (u_i + g_2) + \sum_{\substack{i=a+1 \\ i \neq i_1, i_2}}^{a+b} \frac{\langle K'(x_1), u_i + g_2 K'(x_1) \rangle}{|K'(x_1)|^2}$$

$$+ \frac{\langle K'(x_1), u_{i_3} + u_{i_4} + u_{i_1} + g_2 K'(x_1) \rangle}{|K'(x_1)|^2} + \frac{\langle K'(x_1), u_{i_3} + u_{i_4} + u_{i_2} + g_2 K'(x_1) \rangle}{|K'(x_1)|^2} \Bigg) = 0$$

Thus g_1 and g_2 are simply correction functions which ensure that ϕ preserves the last two requirements of (12).

The automorphism ϕ only affects three of the maps h_{ij}, where the compositions $\phi \circ h_{ij}$ on $\mathcal{S}_{A,B}$ give

$$\phi \circ h_{i_1 i_2} = -h_{i_1 i_2}, \quad \phi \circ h_{i_2 i_3} = -h_{i_2 i_4}, \quad \phi \circ h_{i_2 i_4} = -h_{i_2 i_3}.$$

Therefore ω pulls back to $\mathcal{S}_{A,B}$ as $-\alpha$ (switching the two maps, $h_{i_2 i_3}$ and $h_{i_2 i_4}$, preserves α since this has the effect of switching two $(m-1)$-forms ω_{ij} with m odd).

Also, ϕ does not change the orientation of $\mathcal{S}_{A,B}$. The easiest way to see this is to think of the parameters u_i, and consequently w_i, as numbers and vectors modulo translation and scaling as explained in section §4.1. Functions g_1 and g_2 can then be dropped from the definition of ϕ, whose Jacobian is then easily seen to have positive determinant.

A slight modification is required in the case that one or both of i_3 and i_4 are vertices on the interval. In case i_3 is on the interval, u_{i_3} has to be replaced by $K'(x)u_{i_3}$ in (24)–(29). Same for i_4. The rest of the argument is unchanged, and this completes the proof. □

Lemma 4.6. *Suppose $A \cup B$ contains vertices i_1, i_2, and i_3 with edges between them as in Figure 8. Each of the vertices i_2 and i_3 is either on the interval or free, and the third edge emanating from i_1 ends in a vertex not in $A \cup B$. Then the pushforward of α to $\mathcal{K}$ along $\mathcal{S}_{A,B}$ vanishes.*

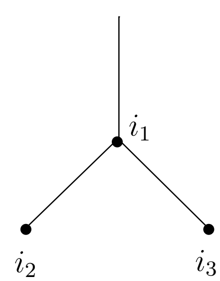


FIGURE 8. The case of Lemma 4.6

Proof. The argument is essentially the same as in the previous lemma. The automorphism ϕ is given by

$$\begin{aligned} u_{i_1} &\longmapsto g_1(u_{i_2} + u_{i_3} - u_{i_1} + g_2 K'(x)), & & \\ u_i &\longmapsto g_1(u_i + g_2), & & i \in \{1, \dots, a\}, \\ u_i &\longmapsto g_1(u_i + g_2 K'(x)), & & i \in \{a+1, \dots, a+b\} \setminus i_i. \end{aligned}$$

As before, we may need to multiply one or both of u_{i_2} and u_{i_3} by $K'(x)$ depending on whether vertices i_2 and i_3 are on the interval or free. Correction functions g_1 and g_2 are again defined so that the parameter constraints are preserved. Composing with the maps h_{ij} gives

$$\phi \circ h_{i_1 i_2} = -h_{i_1 i_3}, \quad \phi \circ h_{i_1 i_3} = -h_{i_1 i_2},$$

and α thus remains unchanged. However, this automorphism reverses the orientation of the stratum, so that the pushforward of α along $\mathcal{S}_{A,B}$ again must be zero. □

Proof of Proposition 4.4. Recall that every trivalent diagram D we are considering is prime, namely not a connected sum of two or more diagrams. It follows that all hidden faces $\mathcal{S}_{A,B}$ of $F[k,s;\mathcal{K},\mathbb{R}^3]$ correspond to subsets of vertices of D with at least one chord or edge connecting a vertex in $A\cup B$ with a vertex not in $A\cup B$. Let v be such a vertex in $A\cup B$ and let e_v be the number of chords or edges connecting it to vertices in the complement of $A\cup B$. Then consider the following cases:

v on interval : In this case, e_v must be 1, and $A\cup B$ is disconnected in the sense of the previous section. The pushforward of α along $\mathcal{S}_{A,B}$ vanishes by Proposition 4.1.

$v\ free,\ e_v=1$: This means that there are two more edges connecting v to vertices in $A\cup B$. But this is precisely the case of Lemma 4.6 so the pushforward is zero.

$v\ free,\ e_v=2$: Now there is another edge emanating from v and ending in a vertex in $A\cup B$. If it ends in an interval vertex, $A\cup B$ is again disconnected. If it ends in a free vertex v', then there are two more edges emanating from this vertex. The situation when both of those end in vertices in $A\cup B$ is precisely the setting of Lemma 4.5. If they both end in vertices not in $A\cup B$, then $A\cup B$ is disconnected. If only one of them ends in a vertex in $A\cup B$, then v' with its edges forms a picture as in Lemma 4.6.

$v\ free,\ e_v=3$: $A\cup B$ is disconnected.

Since there cannot be more than three edges emanating from a vertex, this completes the proof. □

4.4. **Principal faces.** In this situation of exactly two points colliding, there are various cases to consider, depending on whether the vertices in the diagram D are on the interval or free and on whether they are connected by a chord or an edge.

An essential difference from the previous arguments will be that not all the pushforwards will vanish individually, but rather we will have to consider combinations of integrals for various principal faces. The combinations are determined by the STU and IHX relations. We will also now need to pay attention to the labelings of the diagrams.

Recall from Theorem 3.5 that, given a weight system W, the expression claimed to vanish is the derivative

$$dT(W)=\frac{1}{(2n)!}\sum_{D\in TD_n} W(D)(dI(D,K)+dM_D I(D_1,K)), \tag{30}$$

where $dI(D,K)$ is the sum of the pushforwards of α to $\mathcal{K}$ along the faces $\mathcal{S}_{a,b}$ of $F[k,s;\mathcal{K},\mathbb{R}^3]$.

Now consider the three summands determined by labeled diagrams differing only as in the STU relation of Figure 1 in §2. For simplicity, set the labelings as $i=1$ and $j=2$ for now, and we will mention more general labelings later.

The first diagram is associated to $F[k-1,s+1;\mathcal{K},\mathbb{R}^3]$ and the other two to $F[k,s;\mathcal{K},\mathbb{R}^3]$. Each of those has a principal face where points p_1 and p_2, corresponding to vertices 1 and 2 in the three diagrams, come together. Denote these faces by $\mathcal{S}_S$, $\mathcal{S}_T$, and $\mathcal{S}_U$. The goal is to show that the integrals along the three faces have the same value, but with signs as in the STU relation. The sum of the three terms in (30) will then be a multiple of

$$W(S)-W(T)+W(U),$$

and hence 0 by the STU relation.

Let α_S, α_T, and α_U be the three forms which are integrated. It is clear that

$$\int_{\mathcal{S}_T} \alpha_T = -\int_{\mathcal{S}_U} \alpha_U$$

since $\mathcal{S}_T$ and $\mathcal{S}_U$ are diffeomorphic; all the maps h_{ij} for those faces are identical, so that $\alpha_T = \alpha_U$; but the orientations on $\mathcal{S}_T$ and $\mathcal{S}_U$ are different since the two labels, 1 and 2, are switched. The two integrals are thus the same, but with opposite signs since the two vertices are not connected by a chord (had there been a chord between them, the orientation would change, but the map h_{12} would also become its negative, cancelling the negative coming from the orientaton).

Remark. By writing an integral over a face like $\mathcal{S}_T$ and $\mathcal{S}_U$ we mean that the integral is taken over the fiber of that face over $\mathcal{K}$.

It remains to show that, for example,

$$\int_{\mathcal{S}_S} \alpha_S = -\int_{\mathcal{S}_T} \alpha_T.$$

But $\mathcal{S}_S$ is a face of $F[k-1, s+1; \mathcal{K}, \mathbb{R}^3]$, while $\mathcal{S}_T$ is a face of $F[k, s; \mathcal{K}, \mathbb{R}^3]$. There is also an extra map h_{12} on $\mathcal{S}_S$, coming from the edge connecting vertices 1 and 2. However, the remaining maps h_{ij} are the same on the two faces.

To see what the extra map h_{12} is, note that a neighborhood of $\mathcal{S}_S$ is parametrized by

$$(x_1, u, r\,;\, x_2, \ldots, x_{k+s-1}), \quad u \in S^2,\ r \geq 0, \tag{31}$$

with other conditions imposed on the x_i as in (12). The two configuration points corresponding to diagram vertices 1 and 2 are

$$p_1 = K(x_1), \quad p_2 = K(x_1) + ru,$$

so that the extra map on $\mathcal{S}_S$ is simply given by $h_{12} = u$. It follows that α_S and α_T differ only in that α_S contains one more factor than α_T, the $(m-1)$-form $(h_{12})^*\omega_{12}$. In short,

$$\alpha_S = (h_{12})^*\omega_{12} \cdot \alpha_T.$$

But since h_{12} is simply the identity on S^2, $(h_{12})^*\omega_{12}$ may be identified with ω_{12}. More concisely, using Fubini's Theorem, the pushforward of α_S to $\mathcal{K}$ can be rewritten as

$$\int_{\mathcal{S}_S} \alpha_S = \pm \int_{u \in S^2} u^*\omega_{12} \int_{x_i} \alpha_T = \pm \int_{S^2} \omega_{12} \int_{\mathcal{S}_T} \alpha_T = \pm \int_{\mathcal{S}_T} \alpha_T \tag{32}$$

The indeterminacy in sign comes from the possibly different orientations of $\mathcal{S}_S$ and $\mathcal{S}_T$. Seeing that these are in fact opposite is straightforward and we leave this to the reader.

To complete the argument, the parameters for the remaining points in $\mathcal{S}_S$ and $\mathcal{S}_T$ should be mentioned, although they are of no consequence. The outward normal vector giving orientation on these principal faces should always be added to the basis of the tangent space as the last vector. The orientations on $\mathcal{S}_S$ and $\mathcal{S}_T$ will still come out to be different as in the above. This also takes care of an arbitrary labeling of the two vertices considered: Switching any two labels affects the orientations on $\mathcal{S}_S$ and $\mathcal{S}_T$ by switching two tangent vectors or a tangent vector and the outward normal, but the same permutation occurs for both spaces. The orientations on $\mathcal{S}_S$ and $\mathcal{S}_T$ will thus always be different.

The situation is even simpler for the three diagrams in the IHX relation since all three corresponding spaces have the same number of points on and off the knot. It is clear that the integrals of α over $\mathcal{S}_I$, $\mathcal{S}_H$, and $\mathcal{S}_X$ will have the same value, and the labeling of the vertices will ensure that the signs come out as required in the relation.

This leaves the principal faces determined by the pictures in the figure below.

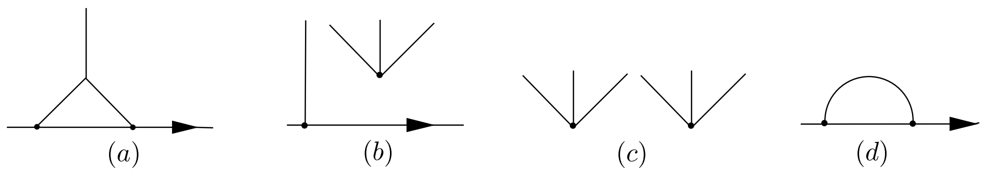


FIGURE 9. Remaining principal faces

When the two vertices on the interval in figure (a) come together, the two maps corresponding to their two edges become the same. Thus h_D on this stratum factors through $(S^2)^{|e|-1}$ and the pullback of ω is thus necessarily 0. If the maps h_{ij} are determined by edges as in figures (b) and (c), the integral vanishes by Proposition 4.1. Finally remember that we are only considering primitive weight systems, namely those that vanish on diagrams that can be obtained as connected sums of two other diagrams. Figure (d) represents such a summand, so that we may disregard this principal face.

4.5. **Faces at infinity.** Here we are saved by the requirement that all our trivalent diagrams are connected, so that the set of trivalent vertices is "anchored" to the interval by at least two edges (it is an easy combinatorial fact that connected trivalent diagrams with free vertices cannot have just one edge connecting the interval vertices to the free ones).

Denote by $\mathcal{S}_\infty$ the stratum with one or more points on the long knot or off it going to infinity.

Proposition 4.7. *The pushforward of α to $\mathcal{K}$ along $\mathcal{S}_\infty$ vanishes.*

Proof. First recall from the end of §4.1 that the parametrization of faces at infinity is much like the one of colliding points, except that r approaches ∞ rather than 0. Next observe that Lemma 4.6 still holds for $\mathcal{S}_\infty$ since the argument used in the proof did not depend on r tending to 0.

Given D, let $\mathcal{S}_\infty$ determined by some subset B of the free vertices of D escaping to infinity. These are connected to the rest of D by some number of external edges. As in the proof of Proposition 4.4, let v be any vertex in B connected to one or more vertices in the complement of B. Also let e_v be the number of the external edges of v. Then we have:

$e_v = 1$: There are two more edges emanating from v which are connected to vertices in B. The picture now is the one of Lemma 4.6 and so the pushforward of α must be 0.

$e_v = 2$: We could invoke Lemma 4.5 to dispense with this case, but a more direct argument is that there will now be two maps h_{ij} which are identical on $\mathcal{S}_\infty$. This is immediate from recalling how h_{ij} extends to strata at infinity, in particular from (16). The pullback $\alpha = (h_D)^*\omega$ to $\mathcal{S}_\infty$ thus factors through a space of lower dimension than that of ω, so that α must be identically 0.

$e_v = 3$: Now there are three maps h_{ij} which are identical, and the comments of the previous case apply again.

This exhausts all the cases since a free vertex has exactly three edges emanating from it. $\square$

4.6. **Anomalous Faces.** The only case of a pushforward along a codimension one stratum of $F[k, s; \mathcal{K}, \mathbb{R}^3]$ left to consider is that of the anomalous face, with all $k + s$ points coming together at the same time. This is the only instance where the situation differs for classical knots and knots in higher codimension. We summarize some of what is known about the anomalous faces and the correction factor $M_D I(D_1, K)$ for classical knots in Proposition 4.8. For $\mathcal{K}_m$, $m > 3$, see Proposition 6.3. The observations and constructions presented here are direct extensions of those in [11].

Recall that the pushforward along the anomalous face vanishes if D has at least one chord (Corollary 4.3). However, this may not be the case for other trivalent diagrams, so assume that D has k interval vertices, s free vertices, and no chords. A neighborhood of the anomalous face $\mathcal{S}_D$ in $F[k, s; \mathcal{K}, \mathbb{R}^3]$ is parametrized by

$$(x, u_1, \ldots, u_k, \ldots, u_{k+s}, r), \tag{33}$$

where $u_1, \ldots, u_k \in I$ and $u_{k+1}, \ldots, u_{k+s} \in \mathbb{R}^3$ satisfy (12).

Since D contains no chords, and all points are colliding, h_D can only be a product of two types of h_{ij} upon restriction to $\mathcal{S}_D$, namely

$$h_{ij} = \frac{u_j - K(x)}{|u_j - K(x)|} \quad \text{and} \quad h_{ij} = \frac{u_j - u_i}{|u_j - u_i|}.$$

There is only one configuration point left after collision, so $\mathcal{S}_D$ maps to $F[1, 0; \mathcal{K}, \mathbb{R}^3]$ by projection on x. Notice now that the pushforward of α may be thought of as

$$\int\limits_{(x,u_i)} \alpha = \int\limits_{x} \Bigg(\int\limits_{\substack{u_i \\ x \text{ fixed}}} \alpha \Bigg).$$

In other words, if we let p be the projection as above, and $\widehat{\pi}$ the bundle map from $F[1, 0; \mathcal{K}, \mathbb{R}^3]$ to $\mathcal{K}$, then the diagram

$$\begin{array}{ccc} \mathcal{S}_D & \xrightarrow{p_*} & \Gamma_{1,0} \\ {\scriptstyle (\partial\pi)_*}\downarrow & \swarrow{\scriptstyle \widehat{\pi}_*} & \\ \mathcal{K} & & \end{array} \tag{34}$$

commutes.

Now let N_D be a subspace of $I^k \times \mathbb{R}^{3s} \times S^2$ consisting of points $(w_1, \dots, w_k, w_{k+1}, \dots, w_{k+s}, v)$ satisfying

(35) $w_1, \dots, w_k \in I$;

(36) $w_{k+1}, \dots, w_{k+s} \in \mathbb{R}^3$;

(37) $w_1 < \cdots < w_k$;

(38) if $i, j > k$ are vertices in D connected by an edge, then $w_i \neq w_j$;

(39) if $i \leq k$, $j > k$ are vertices in D connected by an edge, then $w_i v \neq w_j$;

(40) $\displaystyle\sum_{i=1}^{k+s} |w_i|^2 = 1$;

(41) $\displaystyle\sum_{i=1}^{k} w_i + \sum_{i=k+1}^{k+s} \langle v, w_i \rangle = 0$;

We now get a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_D & \xrightarrow{\ f\ } & N_D \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle g} \\ F[1,0;\mathcal{K},\mathbb{R}^3] & \xrightarrow{\ \tau\ } & S^2 \end{array} \tag{42}$$

where τ is the usual tangential map $x \mapsto K'(x)/|K'(x)|$, g is the projection $(w_i, v) \mapsto v$, and f is given by

$$(x, u_1, \dots, u_k, u_{k+1}, \dots u_{k+s}, 0) \longmapsto (u_1|K'(x)|, \dots, u_k|K'(x)|, u_{k+1}, \dots, u_{k+s}, \frac{K'(x)}{|K'(x)|}).$$

Because of the usual conditions the parameters (33) have to satisfy, f preserves (35)–(41).

Let $\widehat{h}_{ij}$ denote maps from N_D to S^2 determined by the edges of D; if i is on the interval and j is free, or if both are free, we get respectively

$$\widehat{h}_{ij} = \frac{w_j - w_i v}{|w_j - w_i v|}, \quad \widehat{h}_{ij} = \frac{w_j - w_i}{|w_j - w_i|}.$$

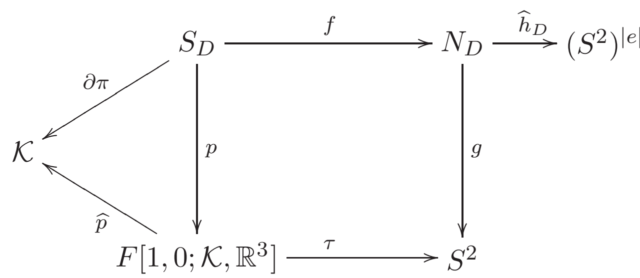
Then h_{ij} clearly factor through N_D, so letting $\widehat{h}_D = \Pi_{edges} \widehat{h}_{ij}$ and $(\widehat{h}_D)^* \omega = \widehat{\alpha}$ gives

$$\alpha = h_D^* \omega = (\widehat{h}_D \circ f)^* \omega = f^*((\widehat{h}_D)^* \omega) = f^* \widehat{\alpha}.$$

Bott and Taubes call $\widehat{\alpha}$ "universal" since it only depends on the diagram D and not on the embedding.

Now use (34) so that $(\partial\pi)_* \alpha = \widehat{p}_*(p_*(f^* \widehat{\alpha}))$. Further, note that N_D was defined precisely so that the diagram in (42) is in fact a pullback square, so we can actually obtain $(\partial\pi)_* \alpha$ by integrating $\widehat{\alpha}$ to S^2, pulling the result back via τ and then integrating to $\mathcal{K}$. In short, the complete diagram we are using is

(43)



with

$$(\partial\pi)_*\alpha = \widehat{p}_*(\tau^*(g_*\widehat{\alpha}), \quad \widehat{\alpha} = (\widehat{h}_D)^*\omega.$$

Since N_D has dimension $k+3s$, the dimension of its fiber over S^2 is $k+3s-2$. On the other hand, $\widehat{\alpha}$ is a $(k+3s)$-form as ω is. It follows that the integration $g_*\widehat{\alpha}$ produces a 2-form on S^2 (remember that $2n = k+s$ is the total number of vertices in D). But this form must be rotationally invariant since ω is. The only such 2-forms on S^2 are constant multiples of the standard unit volume form, which we denote by ω_{12}.

We can now summarize the situation with the anomalous faces in the following

Proposition 4.8. *For a diagram D with chords, the pushforward of α to $\mathcal{K}$ along the anomalous face is zero (Corollary 4.3). For D with no chords, the pushforward is*

$$\mu_D \int\limits_{F[1,0;\mathcal{K},\mathbb{R}^3]} \tau^*\omega_{12}$$

where ω_{12} is the unit volume form on S^2, τ is the tangential map from $F[1,0;\mathcal{K},\mathbb{R}^3]$ to S^2, and μ_D is a real number which depends on D.

Note that this finally proves Theorem 3.5.

To define the correction term $M_D I(D_1, K)$, we now look for a space with boundary $F[1,0;\mathcal{K},\mathbb{R}^3]$ and a map to S^2 which restricts to the tangential map τ on that boundary. The answer can be traced back to §3.1 and is clearly the space $F[2,0;\mathcal{K},\mathbb{R}^3]$ with the normalized difference of the two points, p_1 and p_2 giving the map to the sphere. Note that the boundary of $F[2,0;\mathcal{K},\mathbb{R}^3]$ has two diffeomorphic components, depending on which order p_1 and p_2 appear on the interval.

Definition 4.9. Letting $M_D = \mu_D/2$ and defining $I(D_1, K)$ as before by

$$I(D_1,K) = \int\limits_{F[2,0;\mathcal{K},\mathbb{R}^3]} \left(\frac{p_2-p_1}{|p_2-p_1|}\right)^* \omega_{12} = \int\limits_{F[2,0;\mathcal{K},\mathbb{R}^3]} h_{12}^*\omega_{12}$$

gives the correction term from Theorem 3.5.

This "anomalous term," which has been conjectured to be zero, has been shown to vanish in even degrees [1] and in degrees 3 and 5 [31, 26]. This conjecture has recently been reformulated by S.-W. Yang and C.-H. Yu [37] in terms of the computation of the homology of the trivalent graph complex ($\mathcal{D}$ can be turned into a complex via contraction of edges as the boundary operator). As mentioned in the introduction, the vanishing of the anomaly is intimately related to the question of equivalence of Bott-Taubes integrals and the Kontsevich Integral [15]. Namely, Poirier was able to show that the two are indeed the same up to vanishing of the anomalous term [26]. Using Poirier's work, Lescop [21] additionally proved that the computation of the anomaly only has to be carried out on a certain smaller subclass of diagrams.

5. Universal Finite Type Invariant

One of the most striking features of Chern-Simons perturbation theory is its relation to finite type knot invariants. The fact that configuration space integrals can be used to construct the universal finite type knot invariant has been known for some time [1, 31], and we now provide the details of the proof of this fact. This can be in some sense viewed as completion of Thurston's work in [31]. More details about finite type invariants can be found in [3, 6].

A *singular knot* is a knot as before except for a finite number of double points. The tangent vectors at the double points are required to be independent. A knot with n such self-intersections is called *n-singular*.

Any knot invariant V can be extended to singular knots via a repeated use of the *Vassiliev skein relation* pictured in Figure 10.

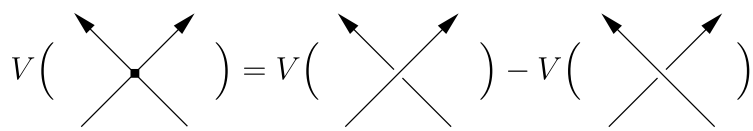


Figure 10. Vassiliev skein relation

The drawings of the knot projections are meant to indicate that the three knots only differ locally in one crossing. The two knots on the right side of the equality are called the *resolutions* of a singularity. A singular knot with n singularities thus produces 2^n resolutions, and the order in which the sigularities are resolved does not matter due to the sign conventions.

Definition 5.1. V is a *(finite, or Vassiliev) type n invariant* if it vanishes identically on singular knots with $n+1$ self-intersections.

Let $\mathcal{V}$ be the collection of all finite type invariants and let $\mathcal{V}_n$ be the type n part of $\mathcal{V}$. An immediate consequence of Definition 5.1 is that $\mathcal{V}_n$ contains $\mathcal{V}_{n-1}$. It is also clear from the definition that $\mathcal{V}_0$ consists only of constant functions on $\mathcal{K}$.

Suppose now that K_1 and K_2 are n-singular knots with singularities in the same place, by which we mean that the points on I which the immersions identify in pairs appear in the same order on I. It is clear that K_2 may be obtained from K_1 by a sequence of crossing changes. But if V is type n, it follows that $V(K_1) = V(K_2)$, since the difference of the value of V on K_1 and its value on the same n-singular knot as K_1 but with a crossing changed is precisely the value of V on an $(n+1)$-singular knot according to the Vassiliev skein relation. Since V is type n, it must vanish on such a knot by definition. We thus note that

The value of a type n invariant on an n-singular knot only depends on the placement of its singularities.

With this observation, it is clear that any type 1 invariant also must be the constant function on $\mathcal{K}$, with the exception of the framing number of one considers framed knots. It can also be shown that, up to framing, there is a unique nontrivial type 2 invariant. In fact, one interpretation of Theorem 5.3 below is that the values of all finite type invariants on all knots can be computed inductively. In practice, however, such computations are quite complicated [3].

Getting back to our observation, the value of a type n invariant V on an n-singular knot thus only depends on what can schematically be represented as an interval with $2n$ paired-off points on it, i.e. a *chord diagram.* The pairs serve as prescriptions for where the singularities on the knot should occur.

Let CD_n be the set of all chord diagrams with n chords. We thus have that a type n invariant V determines a function on CD_n. More precisely, if D is an element of CD_n, and if K_D is any n-singular knot with singularities as prescribed by D, we have a map

$$\mathcal{V}_n \longrightarrow \{f\colon \mathbb{R}[CD_n] \to \mathbb{R}\} \tag{44}$$

given by

$$f(D) = V(K_D) \tag{45}$$

and extending linearly. Note that the kernel is by definition $\mathcal{V}_{n-1}$.

However, not every function on chord diagrams arises in this way: Suppose V is evaluated on a knot with $n-1$ singularities and suppose a strand is moved in a circle around one of them. Along the way, the strand will introduce a singularity each time it passes through another strand emanating from the original singularity. Thus four new n-singular knots are created in the process. Since the strand is back where it started, the sum of the values of V on the four n-singular knots should be zero. But each of the four knots corresponds to a chord diagram. It follows that if f is in the image of the map (44) then the sum of its values on those four diagrams is 0. This is known as the $4T$ (*four-term*) relation, which upon unravelling the four knots into chord diagrams with appropriate signs appear in Figure 11. The diagrams differ only in chords indicated; there may be more chords with their endpoints on the dotted segments, but they are the same for all four diagrams.

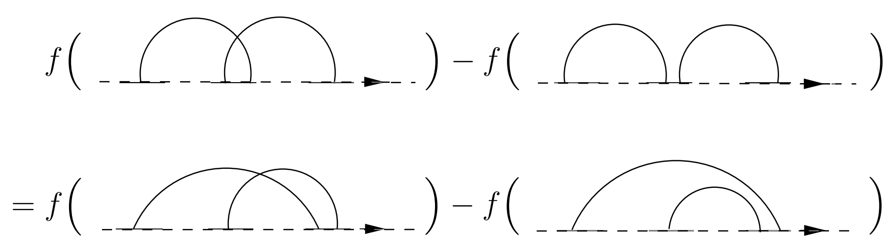


FIGURE 11. $4T$ relation

Remark. In most literature on finite type theory, one more relation besides the $4T$ is imposed. This $1T$ (*one-term) relation* sets the value of any weight system on a chord diagram with an *isolated chord*, i.e. a chord not intersected by any other chords, to be zero. However, if one works with framed knots, $1T$ relation cannot be imposed. This is because the two resolutions of a singularity coming from an isolated chord are *not framed isotopic.* The consequence of having to consider framed knots is simply that the number of finite type invariants is somewhat larger. One now gets a genuine type 1 invariant, the framing number, whose nth power is the additional type n invariant. The anomalous term from Definition 4.9 also has to be modified slightly [26, 22] and varies with the framing.

Let $\mathcal{D}_n^c = \mathbb{R}[CD_n]/4T$ and recall that $\mathcal{D}_n$ is the real vector space generated by trivalent diagrams modulo the STU relation. The following important theorem, due to Bar-Natan, gives the connection between configuration space integrals and finite type invariants:

Theorem 5.2 ([3], Theorem 6)**.** *$\mathcal{D}_n^c$ and $\mathcal{D}_n$ are isomorphic for all positive n, so that every weight system $W \in \mathcal{W}_n$ extends uniquely from $\mathcal{D}_n^c$ to $\mathcal{D}_n$.*

The idea of the proof is to construct the map giving an isomorphism inductively, noting that, in the base case of one free and three interval vertices, the $4T$ relation is the difference of two STU relations.

Now we can think of $\mathcal{W}_n$, the space of weight systems of degree n, as those f in above which vanish on the $4T$ relation. It turns out that $\mathcal{W}_n$ is all there is to the image of the map in (44). Its inverse is given by $T(W)$:

Theorem 5.3. *$T(W)$ is a type n knot invariant. Further, it gives an isomorphism between $\mathcal{W}_n$ and $\mathcal{V}_n/\mathcal{V}_{n-1}$.*

This theorem was first proved by Altschuler and Freidel [1] in a somewhat different form. Before proving it, we need the following

Lemma 5.4. *Let $D' \in CD_n$ be a labeled chord diagram and let $K_{D'}$ be any n-singular knot, $n > 1$, with singularities as prescribed by D'. Also let $K_{D',S}$ be the resolutions of $K_{D'}$ determined by nonempty subsets S of $\{1, \dots, n\}$ where the ith singularity is resolved positively (the first resolution in the Vassiliev skein relation) if $i \in S$. Then, given $\delta > 0$, there are isotopies of the $K_{D',S}$ to knots $K'_{D',S}$ such that*

$$\sum_{K'_{D',S}} M_D I(D_1, K) = 0, \qquad \text{for all labeled } D \in TD_n,\ n > 2 \tag{46}$$

$$\Big| \sum_{K'_{D',S}} M_D I(D_1, K) \Big| < \delta, \qquad \text{for all labeled } D \in TD_2, \tag{47}$$

$$\Big| \sum_{K'_{D',S}} I(D, K'_{D',S}) \Big| < \delta, \qquad \text{for all labeled } D \in TD_n,\ D \neq D'. \tag{48}$$

$$\Big| \sum_{K'_{D',S}} I(D', K'_{D',S}) - 1 \Big| < \delta. \tag{49}$$

Proof. The proof is lengthy but not difficult. We will simply use the fact that we may choose the resolutions so that they differ only inside of n balls in $\mathbb{R}^3$ of arbitrarily small radius.

To show (46), consider the sum of anomalous integrals $M_D I(D_1, K)$ over all resolutions $K_{D',S}$ (with signs according to the skein relation). The resolutions are isotopic to knots which are the same outside of some disjoint balls B_i in $\mathbb{R}^3$, $1 \le i \le n$. In other words, if $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ are points on I which make up the n singularities, the isotopy is given be reparametrizing the interval so that smaller neighborhoods N_i of each b_i are embedded as either the overstrand or the understrand when making the resolutions. All resulting resolutions $K'_{D',S}$ can then be paired off, with opposite signs, into knots differing only in $K'_D(N_i)$. We denote the embedding of N_i as the overstrand by $K'_D(N_i)^+$ and by $K'_D(N_i)^-$ otherwise. Each B_i can be chosen so that it contains both $K'_D(N_i)^+$ and $K'_D(N_i)^-$ and so that it is disjoint from all other B_j. Notice that by arranging a suitable isotopy, B_i can be made to have arbitrarily small radius and contain no other parts of the knot besides the arcs $K'_D(N_i)^+$ and $K'_D(N_i)^-$.

We may now break up $M_D I(D_1, K)$ for each resolution into a sum of integrals over various neighborhoods of the configuration space $F[2, K'_{D',S}]$. These are determined by whether p_1 and p_2 are in some B_i and B_j. If they are both outside of B_i for some i, then the resolutions can be paired off so that each pair differs only inside of B_i. The two integrals in each pair, taken over such a neighborhood, are identical and appear with opposite signs due to the skein relation, so they cancel.

If one or both of p_1 and p_2 are inside some B_i and B_j, a similar situation occurs. Since $n > 2$, we can always pair off the resolutions so that each pair differs only inside of some third ball B_k which contains neither p_1 nor p_2. The integrals again cancel after they are paired according to how they differ inside B_k.

The argument for (47) is more complicated since there are now exactly two balls B_1 and B_2. As p_1 and p_2 may fall into both of them, there are various neighborhoods (depending on whether the points are on the overstrands or the understrands) of $F[2, K'_{D',S}]$ for which the integrals do not cancel. However, we will show that they can be paired so that their difference is arbitrarily small.

For example, pick the resolution $K'_{D',\{1,2\}}$ and a neighborhood of $F[2, K'_{D',\{1,2\}}]$ where p_1 is on the arc $K'_D(N_1)^+$ and the other point, which we call p_2^+, is on $K'_D(N_2)^+$. Then there is another resolution $K'_{D',\{1\}}$ and a neighboorhood of $F[2, K'_{D',\{1\}}]$ such that p_1 is again on the arc $K'_D(N_1)^+$ while the other point, which we now call p_2^-, is on $K'_D(N_2)^-$.

The domains of the integrals $M_D I(D_1, K)$ over this neighborhood are $I \times I$ for every resolution since each arc containing a configuration point is diffeomorphic to I. Further, the integrals appear with opposite signs again due to the skein relation, so that we are now studying the difference

$$\int_{I\times I} \left(\left(\frac{p_2^+ - p_1}{|p_2^+ - p_1|} \right)^* \omega_{12} - \left(\frac{p_2^- - p_1}{|p_2^- - p_1|} \right)^* \omega_{12} \right). \tag{50}$$

The difference of the 2-forms in the integral can be written as

$$\frac{1}{|p_2^+ - p_1|^3}\, det(\dot{p_1}, \dot{p_2}^+, p_2^+ - p_1) - \frac{1}{|p_2^- - p_1|^3}\, det(\dot{p_1}, \dot{p_2}^-, p_2^- - p_1)$$

where the derivatives are taken with respect to the knot parameter t. Now rewrite the above as

$$\begin{aligned} &\frac{1}{|p_2^+ - p_1|^3} \Big(det(\dot{p_1}, \dot{p_2}^+, p_2^+ - p_1) - det(\dot{p_1}, \dot{p_2}^-, p_2^- - p_1) \Big) \\ &\quad + \left(\frac{1}{|p_2^+ - p_1|^3} - \frac{1}{|p_2^- - p_1|^3} \right) det(\dot{p_1}, \dot{p_2}^-, p_2^- - p_1). \end{aligned} \tag{51}$$

To show that the second term can be made small by isotoping the resolutions, we first need to bound the derivatives while the isotopies are performed. To do this, choose smooth bump functions $f(t)$ and $g(t)$ and scale them by

$$f_\rho(t) = \rho^2 f\left(\frac{t}{\rho}\right), \quad g_\rho(t) = \rho^2 g\left(\frac{t}{\rho}\right).$$

The resolutions can be chosen so that the parametrizations for the points are

$$p_2^+ = (t, 0, f_\rho(t)), \quad p_2^- = (t, 0, -f_\rho(t)), \quad p_1 = (t, 0, g_\rho(t))$$

The derivatives $\dot{p_2}^+, \dot{p_2}^-$, and $\dot{p_1}$ are now all bounded so that in particular there is a bound M on the determinant in the second term of (51):

$$det(\dot{p_1}, \dot{p_2}^-, p_2^- - p_1) < M.$$

But the resolutions can also be changed by isotopies so that the distance from p_2^+ to p_2^- is small compared to the distance between p_1 and either of those points. Namely, we can arrange for

$$\frac{1}{|p_2^+ - p_1|^3} - \frac{1}{|p_2^- - p_1|^3} < \frac{\epsilon}{M}.$$

This isotopy shrinks the balls B_1 and B_2.

For the first term in (51), we can use the linearity of the determinant to rewrite it as

$$det(\dot{p_1}, \dot{p_2}^+ - \dot{p_2}^-, p_2^+ - p_1) + det(\dot{p_1}, \dot{p_2}^-, p_2^+ - p_2^-).$$

Here we have that

$$\dot{p_1}, \dot{p_2}^+, \text{ and } \dot{p_2}^- \text{ are bounded because of } f_\delta(t) \text{ and } g_\delta(t),$$
$$\frac{1}{|p_2^+ - p_1|^3} \text{ and } p_2^+ - p_1 \text{ are bounded because of the isotopy, and}$$
$$p_2^+ - p_2^- \text{ is small because of the isotopy.}$$

That leaves the difference

$$\dot{p_2}^+ - \dot{p_2}^- = \left(1, 0, \rho f'\left(\frac{t}{\rho}\right)\right) - \left(1, 0, -\rho f'\left(\frac{t}{\rho}\right)\right).$$

But this difference can be made small through a choice of ρ. Hence the whole first term in (51) can be made smaller than ϵ. Statement (47) follows by choosing $\delta = 2\epsilon$.

As for (48), assume first D is a chord diagram different from D'. Again the resolutions $K_{D',S}$ of an n-singular knot are isotopic to some $K'_{D',S}$ which differ only inside of disjoint balls B_i, $1 \leq i \leq n$, of small radii.

We can break up each configuration space into neighborhoods as before, so that if all the points p_k are outside of some B_i, the integrals cancel in pairs. Otherwise, let U be a neighborhood where each B_i contains at least one point. We then have the following cases:

(1) If not all the points are in the balls, then there exists a p_k in some B_i which is connected to a point p'_k outside of B_i. Assume $p_{k'}$ is either inside another ball B_j or "bounded away" from B_i by another ball as in Figure 12.
For each of such neighborhoods, we can pair off the integrals which differ only inside of B_i. Using the same arguments as in the proof of (47), the differences

$$\left(\frac{p_k^+ - p_{k'}}{|p_k^+ - p_{k'}|}\right)^* \omega_{k'k^+} - \left(\frac{p_k^- - p_{k'}}{|p_k^- - p_{k'}|}\right)^* \omega_{k'k^-} \tag{52}$$

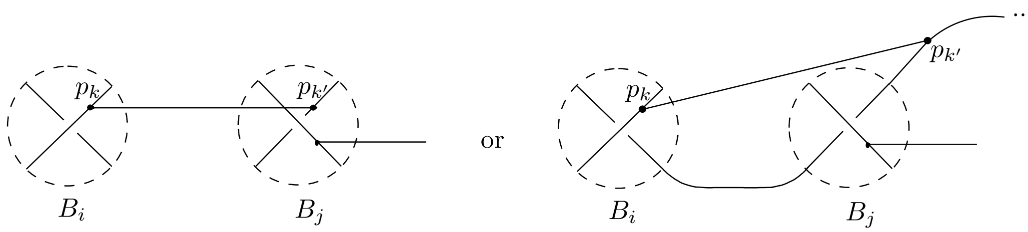


FIGURE 12. Case (1) for D a chord diagram different from D'. A line segment between two points indicates that they are related by a map $h_{p_k p_{k'}}$. We will say the two points are "connected".

can be made small, and we can consequently arrange

$$\Big|\int_U \prod_{\substack{\text{chords } ij\\ ij\neq k'k}} \left(\frac{p_j-p_i}{|p_j-p_i|}\right)^* \omega_{ij} \int_{I\times I} \left(\left(\frac{p_k^+-p_{k'}}{|p_k^+-p_{k'}|}\right)^* \omega_{k'k^+} - \left(\frac{p_k^--p_{k'}}{|p_k^--p_{k'}|}\right)^* \omega_{k'k^-}\right)\Big| < \epsilon \tag{53}$$

for each pair of integrals and for any $\epsilon > 0$. Since there are 2^n resolutions which have been paired off, we have

$$\Big|\sum_{K'_{D',S}} I(D, K'_{D',S})\Big| < 2^{n-1}\epsilon.$$

(2) Assume again p_k is the only point in B_i, but it is connected to a point p'_k near or inside B_i, and the two points are on the same strand as indicated in the left picture of Figure 13. Assume also that either the same situation occurs in every other ball or the ball contains two connected points lying on different strands. Otherwise, we could refer to the situations which have been taken care of already.

For this, isotope the resolutions so that the difference between the overcrossing and the undercrossing in B_i is contained in a ball of smaller radius than that of B_i. The differences (52) can again be made smaller than any positive number by choosing an appropriate isotopy, and the difference in integrals differing in B_i can thus be made to satisfy (53).

(3) Suppose each B_i contains a point p_k which is connected to a point p'_k near or inside B_i. The two points are on different strands, as indicated in the right picture of Figure 13. However, this situation occurs only when $D = D'$.

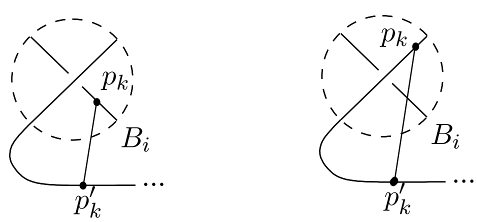


FIGURE 13. Cases (2) and (3)

(4) The the last remaining case is that of all the points in all the balls. Again, because D is different from D', there must be at least two points in different balls which are connected, and we may refer to the first picture in Figure 12. However, before being able to argue as in that case, it has to be noted that there may be more than one difference (52) for each pair of integrals since there may be more than one point p_k in B_i. Consequently, proceeding as we did in that situation will only work if B_i does not contain another pair of connected points on different strands. If this were the case, there would be no way to make the difference of pullbacks small for this map. But we may assume this is not the case since the same cannot happen in B_j (if it did, then there would have to be at least three points in each of B_i and B_j which necessarily means that there is a third ball B_k with no points at all; integrals could be paired according to how they differ in B_k and canceled). We could then pair the integrals which differ only inside of B_j and proceed as before.

Remark. For most neighborhoods of $F[2n, K'_{D',S}]$ with points in the B_i, the integrals in fact cancel. The only cases when this cannot be done is when n of the configuration points are on the arcs $K'_{D',S}(N_i)^+$ for each i, since these are the only arcs in which the resolutions differ . If any one of these arcs is free of configuration points, we can couple the resolutions which differ only in the ball containing that arc and cancel the integrals. Otherwise, (53) is the best that can be done.

Before showing how the sum of the integrals can be made small for trivalent diagrams that are not chord diagrams, we prove the last statement of the lemma. The difference between what was argued so far and this case is that $F[2n, K'_{D',S}]$ now has neighborhoods as in Case (3) above. But no amount of "shrinking" of the B_i will then make the differences (52) small. Changing any of the overcrossings to an undercrossing can be thought of as passing the knot through itself, but as the knot does so, the map changes significantly, regardless of how small the ball B_i is. However, each integral over such a neighborhood is the product

$$\prod_{\text{chords } ij} \int_{S^2_+ \text{ or } S^2_-} \left(\frac{p_j - p_i}{|p_j - p_i|}\right)^* \omega_{ij} = \pm \frac{1}{2^n},$$

where S^2_+ and S^2_- denote the two hemispheres of S^2. The sign depends on how many singularities are resolved negatively (so that there will be as many maps to the lower hemisphere) as well as the labeling of D', which determines the orientation of each $F[2n, K'_{D',S}]$. For example, these always combine to give a positive sign if all singularities are resolved positively.

Now we add the integrals $I(D', K'_{D',S})$ for all the resolutions, but also with negative signs if an odd number of singularities is resolved negatively (from the skein relation). The result is a sum of 2^n terms, each with the value $1/2^n$. Since all other neighborhoods contribute terms that can be made arbitrarily small, we obtain

$$\Big| \sum_{K'_{D',S}} I(D', K'_{D',S}) - 1 \Big| < \delta$$

as desired.

To prove the rest of (48), let D be a trivalent diagram with k interval vertices and $s = 2n - k > 0$ free ones. The integrals are now taken over the fibers of $\Gamma_{k,s} = F[k, s\,; \mathcal{K}, \mathbb{R}^3]$ over $\mathcal{K}$. The fibers are different for each resolution $K'_{D',S}$. However, if we consider the neighborhoods of the fibers with all configuration points on the knot outside of at least one of the balls B_i, the integrals can be paired off according to how they differ in B_i. They will cancel, since the neighborhood chosen is the same for each pair of integrals.

Consider then the neighborhoods in $F[k, s\,; \mathcal{K}'_{D',S}, \mathbb{R}^3]$ for each S such that some or all of the points on the knot are in all of the B_i. Since $k < 2n$, there has to be a B_i with exactly one point p_j on the knot. If there was no such B_i, D would be a chord diagram with $k = 2n$. So p_j is connected to another point $p_{j'}$ outside of B_i. If $p_{j'}$ is on the knot, we are done by the cases considered when D was a chord diagram, and this leaves the case when $p_{j'}$ is off the knot.

Pairing off the resolutions according to how they differ in B_i and then trying to make the differences of integrals $I(D, K'_{D',S})$ small will not work now, since $p_{j'}$ can be anywhere in $\mathbb{R}^3$. But each B_i contains at least one point on the knot. Suppose a of those are associated to at least two. The remaining $n-a$ balls each contain exactly one point on the knot. We have thus far argued that unless all of them are connected to points off the knot, the integrals can be compared in pairs and we are finished. The total number of points accounted for so far is then at least $2a + (n-a) = n + a$. This leaves at most $n-a$ points in $\mathbb{R}^3$ (the total has to be $2n$), and since at least $n+a$ knot points are connected to them, it follows that there has to be at least one point in $\mathbb{R}^3$ connected to points in two of the $n-a$ balls as in Figure 14.

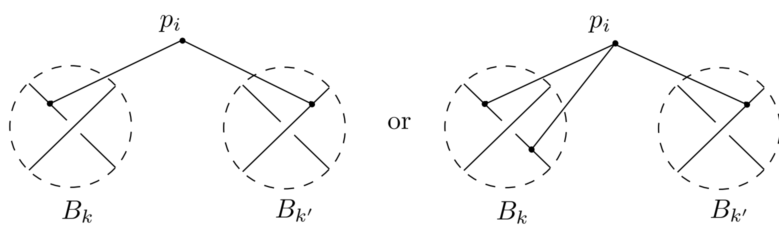


FIGURE 14

For the neighborhood where p_i enters B_k, we can use $B_{k'}$ to pair the resolutions, and vice versa. The neighborhoods are diffeomorphic for all resolutions, so all that remains is to show that the difference of forms which are integrated can be made small. This can be argued exactly the same way as before. The only slight difference is that a point on the knot is replaced by a point in $\mathbb{R}^3$.

If p_i is neither in B_k nor $B_{k'}$, we can enlarge the balls as in Figure 15. Three cases can now be considered: If p_i is in $\widetilde{B}_k \backslash B_k$, $B_{k'}$ can be used for comparing the resolutions. Similarly for p_i in $\widetilde{B}_{k'} \backslash B_{k'}$. If p_i is in the complement of $\widetilde{B}_k \cup \widetilde{B}_{k'}$, either B_k or $B_{k'}$ can be used.

□

Proof of Theorem 5.3. With the above lemma in hand, it is easy to prove that the compositions give the identity. We consider one of them:

$$\mathcal{W}_n \xrightarrow{T(W)} \mathcal{V}_n/\mathcal{V}_{n-1} \hookrightarrow \mathcal{W}_n,$$

where the second map is given by (45). It suffices to show: Given $W \in \mathcal{W}_n$ and a chord diagram $D' \in CD_n$, $W(D') = T(W, K_{D'})$, where

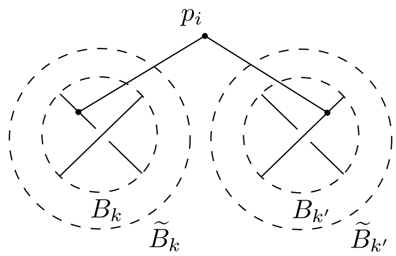


FIGURE 15. The remaining neighborhoods for D a trivalent diagram

$$T(W, K_{D'}) = \frac{1}{(2n)!} \sum_{K_{D',S}} \sum_{\substack{D \in TD_n \\ D \text{ labeled}}} W(D)(I(D, K_{D',S}) - M_D I(D_1, K)). \tag{54}$$

One way to prove this is to show

$$\frac{1}{(2n)!} \sum_{K_{D',S}} \sum_{\substack{D \in TD_n \\ D \text{ labeled}}} D\,(I(D, K_{D',S}) - M_D I(D_1, K)) \;=\; \frac{1}{(2n)!} \Bigg(\sum_{\substack{\text{labelings} \\ \text{of } D'}} (1+\delta) D' + \delta \sum_{K_{D',S}} \sum_{\substack{D \neq D' \\ D \in TD_n \\ D \text{ labeled}}} D \Bigg), \tag{55}$$

where δ can be made arbitrarily small by isotoping the knots $K_{D',S}$. Since $T(W, K_{D'})$ is an invariant, applying W to both sides of (55) will yield that δ must be 0.

A simplification can be made by choosing a definite labeling for each diagram and then showing

$$\sum_{K_{D',S}} \sum_{D \in TD_n} D\,(I(D, K_{D',S}) - M_D I(D_1, K)) \;=\; (1+\delta) D' + \delta \sum_{K_{D',S}} \sum_{\substack{D \neq D' \\ D \in TD_n}} D. \tag{56}$$

Adding over all labelings and dividing by $(2n)!$ will give (55). Moreover, (56) will follow if, given $\delta > 0$, there are isotopies of the resolutions $K_{D',S}$ such that:

$$\Big| \sum_{K_{D',S}} (I(D', K_{D',S}) - M_{D'} I(D_1, K)) - 1 \Big| < \delta$$

$$\Big| \sum_{K_{D',S}} (I(D, K_{D',S}) - M_D I(D_1, K)) \Big| < \delta, \qquad \text{for all } D \neq D'.$$

But this is essentially the statement of Lemma 5.4 for $n > 1$. Noting that in the case $n = 1$ the desired statement is trivially true since there is only one chord diagram with one chord, this completes the proof. □

Remark. The original proof of the isomorphism in Theorem 5.3 is due to Kontsevich [15]. The role of Bott-Taubes integration is played by the Kontsevich Integral mentioned in the introduction.

6. Knots in $\mathbb{R}^m$, $m > 3$

Let $\mathcal{K}_m$ denote the space of knots in $\mathbb{R}^m$, $m > 3$. Bott-Taubes integrals can be defined for $\mathcal{K}_m$ in exactly the same way as described here. This was first done by Cattaneo, Cotta-Ramusino, and Longoni [12] who show that, for a nontrivial weight system $W \in \mathcal{W}_n$, $T(W)$ produces a nontrivial element of $H^{(m-3)n}(\mathcal{K}_m)$. They in fact do more and define a chain complex structure on $\mathcal{D}$ via contraction of edges to prove

Theorem 6.1. *There is a chain map $\mathcal{D} \to \Omega^*(\mathcal{K})$ which induces an injection via Bott-Taubes integration.*

By showing that the diagram complex has nontrivial cohomology in arbitrarily large degrees, one then gets

Corollary 6.2. *For any $k > 0$ and $m > 3$, there exists a $l > k$ with $H^l(\mathcal{K}_m) \neq 0$.*

The main ingredient in the proof of Theorem 6.1 is that Theorem 3.5 does not depend on the dimension of the Euclidean space. To show that, one proceeds in exactly the same way as we have in §4. One only has to be careful with the signs since, depending on whether we are working in $\mathbb{R}^m$ for m even or odd, integrals may change sign if the configuration points are permuted (corresponding to a permutation of the vertices of a diagram) or if the order of those maps is permuted (the chords and edges are permuted). These differences in signs then force a definition of two classes of diagrams [12, 23, 24].

The odd class (the one defined here) differs from the even class in that one also has to label the edges in the even case, and an appropriate sign has to be introduced in the STU relation. The IHX relation of Proposition 2.3 stays the same (with compatibly labeled edges for the three diagrams), but it is not known if the closure relation now holds. This means that it is not clear whether the Hopf algebra $\mathcal{D}$ is commutative in the even case.

With these sign conventions, it is clear that the vanishing and cancelation arguments from §4 go through the same way. One useful observation, however, is that the manifold N_D from §4.6 has dimension $k + ms + m - 3$, so that its fiber dimension over S^{m-1} is $k + ms - 2$. However, $\widehat{\alpha}$ is now a $((k + 3s)(m - 1)/2)$-form. Then $g_*\widehat{\alpha}$ gives a $((m - 3)n + 2)$-form on S^{m-1}. But this form then must be 0 if $m > 3$ for dimensional reasons. Thus we get

Proposition 6.3. *The pushforward of α to $\mathcal{K}_m$, $m > 3$, along the anomalous face vanishes for any diagram.*

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