In search of the perfect knot invariant

Ismar Volić
Wellesley College

MAA Sectional Meeting, Bentley University, 11/22/2008

Plan of talk

Goal: Introduce a class of knot invariants, called *finite type invariants*, which are conjectured to form a "perfect" set of invariants, i.e. they are conjectured to be able to tell all knots apart.

Outline:

1. Introduction to knot theory
2. Some knot invariants and why they are not good enough
3. Introduction to finite type knot invariants
4. My research

Knot theory is central to low-dimensional topology and it has many applications to physics, chemistry, biology, etc.

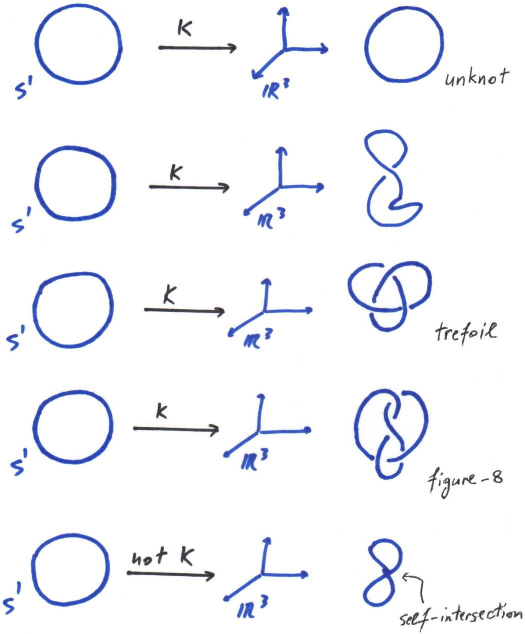
Definition

A *knot* K is a one-to-one, smooth function (embedding) from the circle S^1 to $\mathbb{R}^3$.

Note:

- We usually refer to the *image* of K as the knot.
- We represent knots in the plane by drawing *knot projections* with *strands* and *crossings*. For example,

Intro to knot theory: Examples of knots and not knots



Intro to knot theory: Isotopy

We consider knots up *isotopies*, i.e. deformations that do not tear the knot or pass it through itself. Thus two knots that are isotopic are thought of as the same.

Examples

- The first two knots from the previous slide are isotopic;
- Here is another isotopy:

Intro to knot theory: Invariants

Given two knots, how can we tell (preferably quickly) if they are isotopic or not? For example, are these two knots the same?

$\stackrel{?}{=}$

Definition

A *knot invariant* is a function from the set of knots to some abelian group (usually $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}[x, x^{-1}]$) which gives the same value on isotopic knots.

Note: If an invariant assigns the same value to two knots, that does not necessarily mean that they are isotopic!

Examples of invariants: Constant, tricolorability

Invariant 1: Constant invariant: The function which assigns 0 to every knot is an invariant!

Examples of invariants: Constant, tricolorability

Invariant 1: Constant invariant: The function which assigns 0 to every knot is an invariant!

Invariant 2: Tricolorability: A knot is *tricolorable* if its strands can be colored in one of three colors so that

1. All colors are used
2. At a crossing, either one or three colors meet.

Theorem

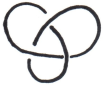
Knots that are tricolorable are not isotopic to those that are not.

Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:

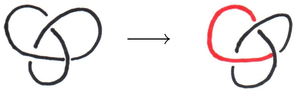
Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



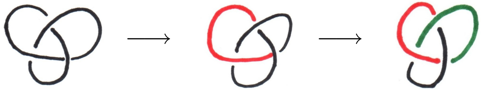
Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



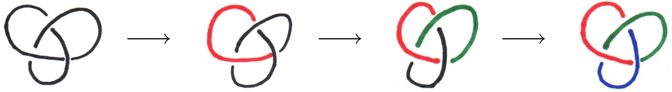
Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



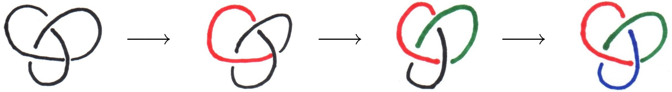
Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



Examples of invariants: Tricolorability cont.

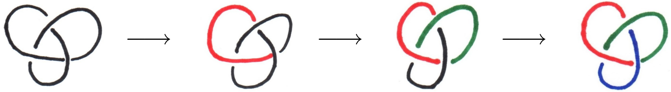
- Trefoil is tricolorable:



- Figure-8 is not tricolorable:

Examples of invariants: Tricolorability cont.

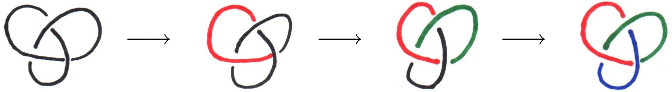
- Trefoil is tricolorable:



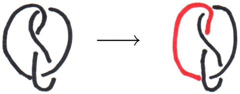
- Figure-8 is not tricolorable:

Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:

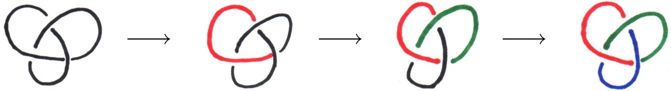


- Figure-8 is not tricolorable:



Examples of invariants: Tricolorability cont.

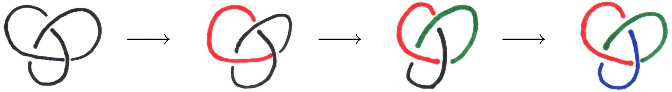
- Trefoil is tricolorable:



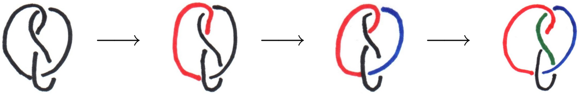
- Figure-8 is not tricolorable:

Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:

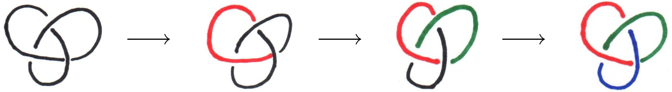


- Figure-8 is not tricolorable:

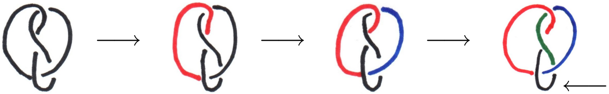


Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



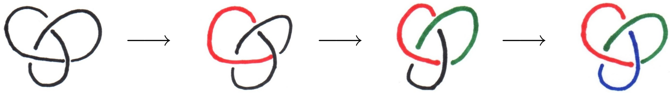
- Figure-8 is not tricolorable:



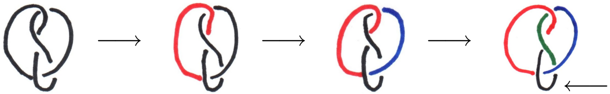
Cannot color this strand so that three colors meet at all crossings

Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



- Figure-8 is not tricolorable:

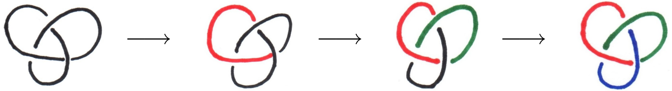


Cannot color this strand
so that three colors
meet at all crossings

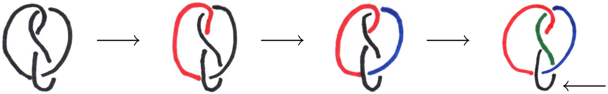
- Unknot is not tricolorable (no way to use three colors).

Examples of invariants: Tricolorability cont.

- Trefoil is tricolorable:



- Figure-8 is not tricolorable:



Cannot color this strand
so that three colors
meet at all crossings

- Unknot is not tricolorable (no way to use three colors).

So we have

$$\boxed{\text{unknot} \neq \text{trefoil} \neq \text{figure-8}}$$

but

$$\boxed{\text{unknot} \stackrel{?}{=} \text{figure-8}}$$

We don't know! We need a different invariant.

Invariant 3: Jones polynomial:

Polynomial invariants caused a resurgence of interest in knot theory in the '60s. There are many polynomial invariants today: Jones, HOMFLY, Alexander, Conway, etc. which are fairly efficient at telling knots apart.

To define the Jones polynomial of a knot K, first compute the *bracket polynomial* $\langle K \rangle$ which is a Laurent polynomial in variable X defined by the rules

1. $\langle$  $\rangle = 0$
2. $\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$
3. $\langle$knot $\cup$  $\rangle = (-X^2 - X^{-2})\langle$knot$\rangle$

Examples of invariants: Jones polynomial cont.

Example

$\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$

$= X\langle$  $\rangle + X^{-1}\Big((-X^2 - X^{-2})\langle$  $\rangle\Big)$

$= -X^{-3}\langle$  $\rangle = -X^{-3}$

Examples of invariants: Jones polynomial cont.

Example

$\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$

$= X\langle$  $\rangle + X^{-1}\Big((-X^2 - X^{-2})\langle$  $\rangle\Big)$

$= -X^{-3}\langle$  $\rangle = -X^{-3}$

Example

$\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$

$= X\Big(X\langle$  $\rangle + X^{-1}\langle$  $\rangle\Big) + X^{-1}\langle$  $\rangle$

$= X\Big(X\langle$  $\rangle + X^{-1}(-X^2 - X^{-2})\langle$  $\rangle\Big) + X^{-1}\langle$  $\rangle$

$= \cdots = X^7 - X^3 - X^{-5}$

Examples of invariants: Jones polynomial cont.

Jones polynomial is then defined as

$$\boxed{J(K) = (-X^3)^{-w(K)}\langle K\rangle}$$

where $w(K)$ is the number of crossings of K, counted properly.

This is a knot invariant. Its value on the figure-8 knot happens to be a non-trivial polynomial, so now we know

$$\boxed{\text{figure-8} \neq \text{unknot}}$$

However, there exist two knots which we know are different and whose Jones polynomials are the same. So Jones polynomial cannot tell all knots apart.

This is the case with *all* invariants (and in fact all sets of invariants) we know. This leads to one of the two most important open questions in knot theory:

Main open questions in knot theory

Two of the biggest open questions in knot theory are:

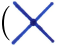
1. Can we enumerate all knot types, i.e. all equivalence classes of knots under the relation of isotopy?
   (In other words, what is H_0(space of knots)?)

2. Is there a *complete* invariant or a set of invariants? That is, is there a set of invariants (possibly infinite) such that, given any two knots that are not isotopic, there is an invariant in this set that can tell them apart?
   (In other words, what is H^0(space of knots)?)

The second question (which is what we have been discussing so far) is what some of my research is concerned with.

Possible answer to 2nd question: Finite type invariants

A set of invariants that is conjectured to separate knots is the set of *finite type invariants* defined as:
Given a knot invariant V, extend it to *singular knots*, i.e. knots with finite number of self-intersections via the *Vassiliev skein relation*

$V($  $)$ $=$ $V($  $)$ $-$ $V($  $)$

Resolving n singularities one by one (order does not matter), we get 2^n knots on which we can evaluate V.

Definition

An invariant V is *finite type (or Vassiliev of type) n* if it vanishes on all singular knots with $n+1$ singularities.

Examples

- Coefficients of various polynomial invariants
- Milnor invariants

Finite type invariants cont.

Some facts about finite type theory:

Finite type invariants cont.

Some facts about finite type theory:

- Finite type invariants have been studied extensively in the past 10 years because of their many connections to physics (Chern-Simons theory, Feynman diagrams), 3-manifold theory, Lie algebras, etc.

Finite type invariants cont.

Some facts about finite type theory:

- Finite type invariants have been studied extensively in the past 10 years because of their many connections to physics (Chern-Simons theory, Feynman diagrams), 3-manifold theory, Lie algebras, etc.
- They are isomorphic to (the dual of) a certain algebra of chord diagrams and are very combinatorial in nature.

Finite type invariants cont.

Some facts about finite type theory:

- Finite type invariants have been studied extensively in the past 10 years because of their many connections to physics (Chern-Simons theory, Feynman diagrams), 3-manifold theory, Lie algebras, etc.
- They are isomorphic to (the dual of) a certain algebra of chord diagrams and are very combinatorial in nature.
- Better yet, they can be *constructed* from chord diagrams using various integration techniques (*Kontsevich Integral, Bott-Taubes configuration space integrals*).

Finite type invariants cont.

Some facts about finite type theory:

- Finite type invariants have been studied extensively in the past 10 years because of their many connections to physics (Chern-Simons theory, Feynman diagrams), 3-manifold theory, Lie algebras, etc.
- They are isomorphic to (the dual of) a certain algebra of chord diagrams and are very combinatorial in nature.
- Better yet, they can be *constructed* from chord diagrams using various integration techniques (*Kontsevich Integral, Bott-Taubes configuration space integrals*).
- Finite type invariants can be defined for links, homotopy links, and braids. They in fact *separate homotopy links and braids*, which is good evidence that they might also separate knots and ordinary links.

My work

My research is in *calculus of functors* and in particular in how it applies in the case of the space/functor $Emb(M, N)$ of embeddings of a manifold M in a manifold N (special case of this is classical knot theory, $Emb(S^1, \mathbb{R}^3)$).

Since "*Emb*" can be thought of as a functor of open subsets of M, one can apply *manifold calculus of functors*. If N is a vector space, then one can also apply *orthogonal calculus of functors*.

The ultimate goal is to understand the homotopy type of the space $Emb(M, N)$ by breaking it up into simpler pieces by applying these two variants of functor calculus.

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

- Given a complete description of the rational homotopy and (co)homology of the space of knots $Emb(S^1, \mathbb{R}^n)$ for $n > 3$.

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

- Given a complete description of the rational homotopy and (co)homology of the space of knots $Emb(S^1, \mathbb{R}^n)$ for $n > 3$.
  - Extended Kontsevich's result about rational formality of the little cubes operad.
  - Developed rational homotopy theory of semi-algebraic spaces.

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

- Given a complete description of the rational homotopy and (co)homology of the space of knots $Emb(S^1, \mathbb{R}^n)$ for $n > 3$.
  - Extended Kontsevich's result about rational formality of the little cubes operad.
  - Developed rational homotopy theory of semi-algebraic spaces.
- By specializing to $H^0(Emb(S^1, \mathbb{R}^3))$, discovered finite type theory in calculus of functors and recast the separation conjecture in that language (where it seems more tractable). The goal is to reprove that finite type invariants separate braids and homotopy links using calculus of functors and to try to extend the techniques of those proofs to knots and links.

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

- Given a complete description of the rational homotopy and (co)homology of the space of knots $Emb(S^1, \mathbb{R}^n)$ for $n > 3$.
  - Extended Kontsevich's result about rational formality of the little cubes operad.
  - Developed rational homotopy theory of semi-algebraic spaces.
- By specializing to $H^0(Emb(S^1, \mathbb{R}^3))$, discovered finite type theory in calculus of functors and recast the separation conjecture in that language (where it seems more tractable). The goal is to reprove that finite type invariants separate braids and homotopy links using calculus of functors and to try to extend the techniques of those proofs to knots and links.
- Discovered that the rational homology of $Emb(M, \mathbb{R}^n)$ only depends on the rational homotopy type of M in a range of dimensions.

My work cont.

Here are some of the things I have done or am currently working on (often with various subsets of my collaborators – G. Arone (UVA), P. Lambrechts (Louvain), V. Turchin (KSU), R. Hardt (Rice), B. Munson (Wellesley)):

- Given a complete description of the rational homotopy and (co)homology of the space of knots $Emb(S^1, \mathbb{R}^n)$ for $n > 3$.
  - Extended Kontsevich's result about rational formality of the little cubes operad.
  - Developed rational homotopy theory of semi-algebraic spaces.
- By specializing to $H^0(Emb(S^1, \mathbb{R}^3))$, discovered finite type theory in calculus of functors and recast the separation conjecture in that language (where it seems more tractable). The goal is to reprove that finite type invariants separate braids and homotopy links using calculus of functors and to try to extend the techniques of those proofs to knots and links.
- Discovered that the rational homology of $Emb(M, \mathbb{R}^n)$ only depends on the rational homotopy type of M in a range of dimensions.
- Currently developing *multivariable* calculus of functors and extending Milnor invariants of homotopy links to other link spaces.