Topology and the problem of telling knots apart

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Outline of the talk

1. Some generalities about mathematics
2. Brief introduction to topology
3. Brief introduction to knot theory
4. Some examples of knot invariants and why they are not good enough
5. Finite type knot invariants and why they might be good enough
6. Back to generalities about mathematics

1. Mathematics: subdivision

The coarsest unofficial subdivision of mathematics is into:

1. Logic and foundations
2. Algebra
3. Number theory
4. Geometry
5. Topology
6. Real analysis
7. Complex analysis
8. Functional analysis
9. Numerical analysis
10. Differential equations
11. Probability and statistics
12. Applied mathematics

My research field is Topology.

1. Mathematics: subdivision cont.

The coarsest official subdivision of mathematics is into:

1. General
2. History and biography
3. Mathematical logic and foundations
4. Combinatorics
5. Order, lattices, ordered algebraic structures
6. General algebraic systems
7. Number theory
8. Field theory and polynomials
9. Commutative rings and algebras
10. Algebraic geometry
11. Linear and multilinear algebra; matrix theory
12. Associative rings and algebras
13. Nonassociative rings and algebras
14. Category theory; homological algebra
15. K-theory
16. Group theory and generalizations
17. Topological groups, Lie groups
18. Real functions
19. Measure and integration
20. Functions of a complex variable
21. Potential theory
22. Several complex variables and analytic spaces
23. Special functions
24. Ordinary differential equations
25. Partial differential equations
26. Dynamical systems and ergodic theory
27. Difference and functional equations
28. Sequences, series, summability
29. Approximations and expansions
30. Fourier analysis
31. Abstract harmonic analysis
32. Integral transforms, operational calculus
33. Integral equations
34. Functional analysis
35. Operator theory
36. Calculus of variations and optimal control; optimization
37. Geometry
38. Convex and discrete geometry
39. Differential geometry
40. General topology
41. Algebraic topology
42. Manifolds and cell complexes
43. Global analysis, analysis on manifolds
44. Probability theory and stochastic processes
45. Statistics
46. Numerical analysis
47. Computer science
48. Mechanics of particles and systems
49. Mechanics of deformable solids
50. Fluid mechanics
51. Optics, electromagnetic theory
52. Classical thermodynamics, heat transfer
53. Quantum theory
54. Statistical mechanics, structure of matter
55. Relativity and gravitational theory
56. Astronomy and astrophysics
57. Geophysics
58. Operations research, mathematical programming
59. Game theory, economics, social and behavioral sciences
60. Biology and other natural sciences
61. Systems theory; control
62. Information and communication, circuits
63. Mathematics education

1. Mathematics: subdivision cont.

Fields I use/contribute to in my research:

1. General
2. History and biography
3. Mathematical logic and foundations
4. Combinatorics
5. Order, lattices, ordered algebraic structures
6. General algebraic systems
7. Number theory
8. Field theory and polynomials
9. Commutative rings and algebras
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11. Linear and multilinear algebra; matrix theory
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42. **Manifolds and cell complexes**
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62. Information and communication, circuits
63. Mathematics education

1. Mathematics: subdivision cont.

- Mathematics is further subdivided into over 5,000 subjects.
- Every paper is identified by one (or more) of these subjects. For example,

>
>
> s *Subject Classification.* Primary: 55P62; Secondary: 18D50.
> *ırases.* Operad formality, little cubes operad, Fulton-MacPherson operad.
> is Chercheur Qualifié au F.N.R.S. The second author was supported in part by the
> ındation grant DMS 0504390.
>
> 1

55: Algebraic Topology (one of the 63 top-level categories from previous slide)

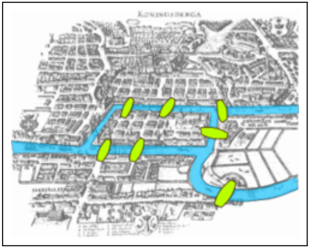
P: Homotopy Theory

62: Rational Homotopy Theory

Topology: history

Topology is the study of topological spaces, which are sets with extra structure, and functions between them.

It began with Euler's 1736 solution of the *Seven bridges of Königsberg* problem:



Is there a way to cross the seven bridges in the picture only once and come back to the starting point?
Answer: No.

Modern topology starts with the famous 1904 *Poincaré Conjecture*:
If it "looks" like a sphere in 4-dimensional Euclidean space, is it a sphere?
(Recently solved in the affirmative by Grigori Perelman.)

Topology: homeomorphism

The main difference between geometry and topology is that shape is important in geometry, while this is not the case in topology. This is why topology is often called *rubber-sheet geometry*.

Definition

Official: A *homeomorphism* $h\colon X \to Y$ between topological spaces X and Y is a continuous bijection with continuous inverse. If there exists a homeomorphism between two spaces, they are said to be *homeomorphic*.

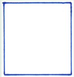
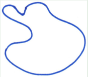
Unofficial: Two spaces are *homeomorphic* if one can be deformed into the other without cutting, tearing, or poking holes (or other violent acts).

(As we will see soon, the latter is just an intuitive definition and is *not always right*, but it is good enough for many purposes.)

If X and Y are homeomorphic, we consider them to be the same and simply write $X = Y$.

Topology: examples of homeomorphisms

Example

 $=$  $\neq$ 

square $=$ closed curve $\neq$ non-closed curve

Example

Which letters are homeomorphic?

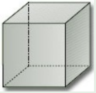
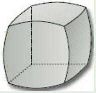
A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z

A=R,
C=G=I=J=L=M=N=S=U=V=W=Z,
D=O,
E=F=T=Y,
H=K,

B, P, Q, and X are not homeomorphic to any other letters (in this font).

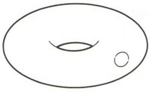
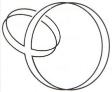
Topology: examples of homeomorphisms cont.

Example

 =  = 

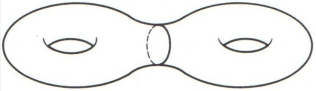
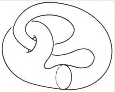
cube = sphere

Example

 = 

punctured torus (surface of a donut) = two ribbons glued together

Example

 = 

double torus = double torus with arms linked

Topology: examples of homeomorphisms cont.

Example

coffee mug = donut

Topology: examples of homeomorphisms cont.

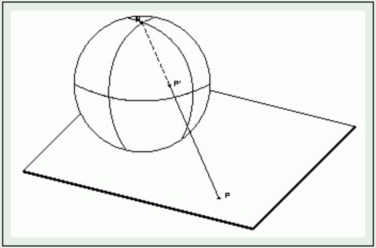
Example (Stereographic projection)

Let $S^n \setminus \{\text{north pole}\}$ be the n-dimensional sphere missing a point and let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The two are homeomorphic. Here is the homeomorphism for $n = 2$:

$$h\colon \mathbb{R}^2 \longrightarrow S^2 \setminus \{\text{north pole}\}$$

$$(x, y) \longmapsto \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)$$

The picture is:



Topology: examples of homeomorphisms cont.

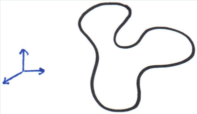
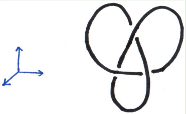
Example

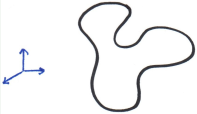
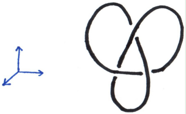
$$6 \overset{?}{=} 9 \quad 6 = 9$$

Example

$$4 \overset{?}{=} 4$$
$$4 \neq 4$$

Example

 $\overset{?}{=}$ 

 $=$ 

Intro to knot theory: Definition of a knot

Knot theory is central to low-dimensional topology and it has many applications to physics, chemistry, biology, etc.

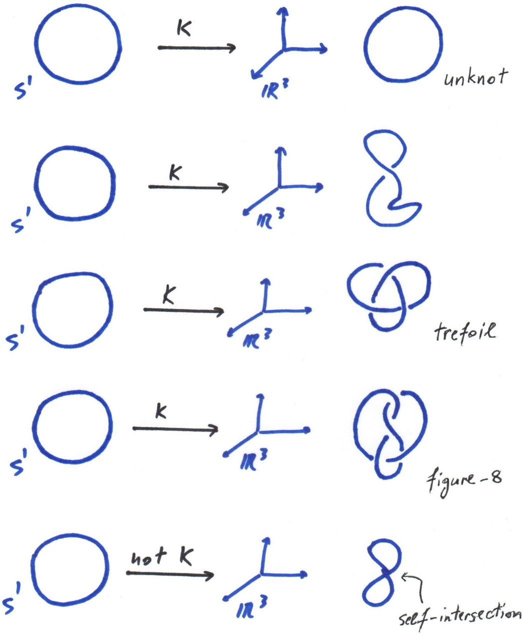
Definition

A *knot* K is a one-to-one, smooth function (embedding) from the circle S^1 to $\mathbb{R}^3$.

Note:

- We usually refer to the *image* of K as the knot.
- We represent knots in the plane by drawing *knot projections* with *strands* and *crossings*. For example,

Intro to knot theory: Examples of knots and not knots



Intro to knot theory: Isotopy

We consider knots up to *isotopies*, which are deformations that do not cut the knot *or pass it through itself* (so this is stronger than homeomorphism and precisely what we want to avoid the situation from an earlier example). Two knots that are isotopic are thought of as the same.

Examples

- The first two knots from the previous slide are isotopic;
- Here is another isotopy:

Intro to knot theory: Invariants

Given two knots, how can we tell (preferably quickly) if they are isotopic or not? For example, are these two knots the same?

$\stackrel{?}{=}$

Definition

A *knot invariant* is a function which assigns to each knot a number or a polynomial (or possibly a more complicated object) and which gives the same value on isotopic knots.

Note:

- This is most useful in the contrapositive.
- If an invariant assigns the same value to two knots, that does not necessarily mean that they are isotopic!

Examples of knot invariants: Constant, tricolorability

Invariant 1. Constant invariant: The function which assigns 0 to every knot is an invariant!

Invariant 2. Tricolorability: A knot is *tricolorable* if its strands can be colored in one of three colors so that

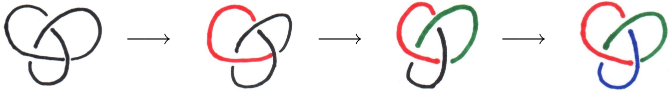
1. All colors are used
2. At a crossing, either one or three colors meet.

Theorem

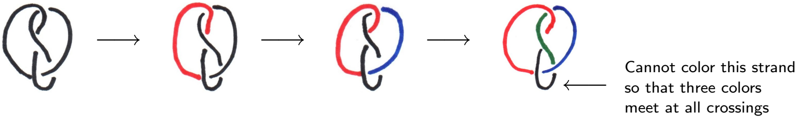
Knots that are tricolorable are not isotopic to those that are not.

Examples of knot invariants: Tricolorability cont.

- Trefoil is tricolorable:



- Figure-8 is not tricolorable:



- Unknot is not tricolorable (no way to use three colors).

So we have

$$\boxed{\text{unknot} \neq \text{trefoil} \neq \text{figure-8}}$$

but

$$\boxed{\text{unknot} \stackrel{?}{=} \text{figure-8}}$$

We don't know! We need a different invariant.

Examples of knot invariants: Jones polynomial

Invariant 3. Jones polynomial:

Polynomial invariants, which to each knot assign a polynomial, caused a resurgence of interest in knot theory in the '60s. There are many polynomial invariants today: Jones, HOMFLY, Alexander, Conway, etc. which are fairly efficient at telling knots apart.

The Jones polynomial of a knot K, denoted by $\langle K \rangle$, is a polynomial in positive and negative powers of a variable X, defined by the rules

1. $\langle$  $\rangle = 0$
2. $\langle$  $\rangle = X \langle$  $\rangle + X^{-1} \langle$  $\rangle$
3. $\langle$knot $\cup$  $\rangle = (-X^2 - X^{-2})\langle$knot$\rangle$

Examples of knot invariants: Jones polynomial cont.

Example

$\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$

$= X\langle$  $\rangle + X^{-1}\Big((-X^{2} - X^{-2})\langle$  $\rangle\Big)$

$= -X^{-3}\langle$  $\rangle = -X^{-3}$

Example

$\langle$  $\rangle = X\langle$  $\rangle + X^{-1}\langle$  $\rangle$

$= X\Big(X\langle$  $\rangle + X^{-1}\langle$  $\rangle\Big) + X^{-1}\langle$  $\rangle$

$= X\Big(X\langle$  $\rangle + X^{-1}(-X^{2} - X^{-2})\langle$  $\rangle\Big) + X^{-1}\langle$  $\rangle$

$= \cdots = X^{7} - X^{3} - X^{-5}$

Examples of knot invariants: Jones polynomial cont.

Jones polynomial is (almost) a knot invariant, i.e. isotopic knots have same Jones polynomial. The Jones polynomial of the figure-8 knot is $1 - X^2$. But since $\langle \text{unknot} \rangle = 0$ by definition, it follows that

$$\boxed{\text{figure-8} \neq \text{unknot}}$$

However, there exist two (fairly complicated) knots which we know are not isotopic by other methods and whose Jones polynomials are the same. So Jones polynomial cannot tell all knots apart!

This is the case with *all* known invariants (and in fact all sets of invariants) we know. This leads to one of the two most important open questions in knot theory:

Main open questions in knot theory

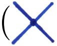
Two of the biggest open questions in knot theory are:

1. Can we enumerate all knot types, i.e. all equivalence classes of knots under the relation of isotopy?
2. Is there a *complete* invariant or a set of invariants? That is, is there a set of invariants (possibly infinite) such that, given any two knots that are not isotopic, there is an invariant in this set that can tell them apart?

The second question (which is what we have been discussing so far) is what some of my research is concerned with.

Possible answer to 2nd question: Finite type invariants

A set of invariants that is conjectured to be complete is the set of *finite type invariants* defined as:
Given a knot invariant V, extend it to *singular knots*, i.e. knots with finite number of self-intersections via the *Vassiliev skein relation*

$V($  $) = V($  $) - V($  $)$

Resolving n singularities one by one (order does not matter), we get 2^n knots on which we can evaluate V.

Definition

An invariant V is *finite type (or Vassiliev of type) n* if it vanishes on all singular knots with $n+1$ singularities.

Examples

- (Coefficients of) various polynomial invariants
- Milnor invariants

Finite type invariants cont./my work

Some facts about finite type theory:

- Finite type invariants have been studied extensively in the past 10 years because of their many connections to physics (Chern-Simons theory, Feynman diagrams), 3-manifold theory, Lie algebras, etc.
- They are very combinatorial in nature and can in fact be constructed from certain kinds of diagrams using various integration techniques (Kontsevich Integral, Bott-Taubes configuration space integrals).
- Finite type invariants do form a complete set of invariants for certain kinds of links (two or more knotted-up knots) which is good evidence that they might form a complete set of invariants for knots as well.

In my work, I study generalizations of the space of knots and use a theory called *calculus of functors* to study their topological structure. In particular, one of the questions I am interested in is whether finite type invariants form a complete set of invariants.

Paper sample

THE RATIONAL HOMOLOGY OF SPACES OF LONG KNOTS IN CODIMENSION >2

PASCAL LAMBRECHTS, VICTOR TURCHIN, AND ISMAR VOLIĆ

ABSTRACT. We determine the rational homology of the space of long knots in $\mathbb{R}^d$ for $d \geq 4$. Our main result is that the Vassiliev spectral sequence computing this rational homology collapses at the E^1 page. As a corollary we get that the homology of long knots (modulo immersions) is the Hochschild homology of the Poisson algebras operad with bracket of degree $d-1$, which can be obtained as the homology of an explicit graph complex and is in theory completely computable.

Our proof is a combination of a relative version of Kontsevich's formality of the little d-disks operad and of Sinha's cosimplicial model for the space of long knots arising from Goodwillie-Weiss embedding calculus. As another ingredient in our proof, we introduce a generalization of a construction that associates a cosimplicial object to a multiplicative operad. Along the way we also establish some results about the Bousfield-Kan spectral sequences of a truncated cosimplicial space.

1. Introduction

A *long knot* is a smooth embedding $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$, $d \geq 3$, that coincides with a fixed linear embedding $\epsilon\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ outside a compact set. We denote by $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ the space of all long knots equipped with the weak C^∞ topology. This space, and in particular its homology, has been under investigation for more than twenty years. One of the first important tools in this study was a spectral sequence for computing the homology of spaces of knots constructed by Vassiliev in the late eighties [36]. This spectral sequence sparked a lot of interest, especially in dimension $d = 3$, where it is related to the theory of finite type knot invariants.

Independently, Goodwillie and Weiss suggested another approach for the study of knots and more general spaces of embeddings which is now known as *embedding calculus*. In particular, it is suggested in [15, Example 5.1.4] that this approach should also give a spectral sequence for computing the homology of spaces of long knots. Indeed, it later turned out that it does, and that this spectral sequence was equivalent to Vassiliev's. Goodwillie-Weiss embedding calculus for knots was developed further by Sinha in [29], who also emphasized the connection with the little d-disks operad in [31].

Vassiliev spectral sequence arises from a study of the *discriminant set*, i.e. the complement of the set of knots in the space of all smooth maps $\mathbb{R} \to \mathbb{R}^d$ with fixed behavior at infinity (or more precisely in some finite-dimensional approximation of that space). In other words, the discriminant set is the set of maps with singularities. This set admits a nice stratification, which in turn yields a natural filtration from which the spectral sequence is constructed. A classification of the singularities gives the E^1 page of this spectral sequence a combinatorial description in terms of certain graphs such as chord diagrams familiar from finite type knot theory. When $d \geq 4$, this spectral sequence converges to the homology of the space of long knots [36, Section 6.6].

1991 *Mathematics Subject Classification.* Primary: 57Q45; Secondary: 55P62, 57R40.

Key words and phrases. knot spaces, embedding calculus, formality, operads, Bousfield-Kan spectral sequence.

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Paper sample

Vassiliev has conjectured a stable splitting of the resolved discriminant which would imply that his spectral sequence collapses at the E^1 page [39]. This collapse was proved rationally by Kontsevich in dimension $d = 3$ along the diagonal $E^1_{-p,p}$. The proof uses the famous *Kontsevich Integral*, a map that realizes all finite type invariants [16]. Kontsevich further claimed in [17, Theorem 2.3] that his integration approach can be generalized for $d \geq 4$ to give a proof of the collapse of the rational Vassiliev spectral sequence everywhere. In [7], Cattaneo, Cotta-Ramusino, and Longoni filled in some details of that program and proved the collapse along the main diagonal. As far as we know, however, no complete proof of the rational collapse has yet appeared.

In this paper, we give a proof of Vassiliev's conjecture over the rationals.

Theorem 1.1. *For $d \geq 4$, the Vassiliev spectral sequence computing the rational homology of the space of long knots* $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ *collapses at the E^1 page.*

To prove this theorem, we take a very different approach than that of Kontsevich, namely the Goodwillie-Weiss calculus of the embedding functor and Sinha's cosimplicial model for spaces of knots arising from this theory. Before explaining this further, it is convenient to introduce a variation of the space of long knots. Consider first the *space of long immersions*

$$\mathrm{Imm}(\mathbb{R}, \mathbb{R}^d) := \{f : \mathbb{R} \looparrowright \mathbb{R}^d : f \text{ is an immersion that coincides with } \epsilon \text{ outside a compact set}\}.$$

There is an inclusion $\iota : \mathrm{Emb}(\mathbb{R}, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}(\mathbb{R}, \mathbb{R}^d)$ and its homotopy fiber, $\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d)$, is called the *space of long knots modulo immersions*. By Smale-Hirsch theory [32], there is a weak equivalence $\mathrm{Imm}(\mathbb{R}, \mathbb{R}^d) \simeq \Omega S^{d-1}$. Moreover ι is null-homotopic [31, Proposition 5.17], so we have a weak equivalence

$$\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{Emb}(\mathbb{R}, \mathbb{R}^d) \times \Omega^2 S^{d-1}. \tag{1.1}$$

In [31] Sinha constructs a cosimplicial space

$$\mathcal{K}_d^\bullet = \big(\mathcal{K}_d(0) \rightrightarrows \mathcal{K}_d(1) \rightrightarrows\!\!\!\rightarrow \mathcal{K}_d(2) \cdots\big),$$

where $\mathcal{K}_d = \{\mathcal{K}_d(n)\}_{n \geq 0}$ is a topological operad homotopy equivalent to the little d-disks operad, called the *Kontsevich operad*. This operad turns out to be *multiplicative*, i.e. there exists a map $\{*\}_{n \geq 0} \to \mathcal{K}_d$ from the nonsymmetric associative topological operad which consists of the one-point space in each degree, to the Kontsevich operad. The cofaces and codegeneracies of $\mathcal{K}_d^\bullet$ are induced from this multiplicative structure via a general construction of Gerstenhaber and Voronov [12] which we recall in Section 2.2 using the McClure-Smith point of view [26, Section 3]. The main result of [31] is that for $d \geq 4$ the homotopy totalization of that cosimplicial space, $\mathrm{hoTot}(\mathcal{K}_d^\bullet)$, is weakly equivalent to $\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d)$.

The homology Bousfield-Kan spectral sequence [5] associated to $\mathcal{K}_d^\bullet$ converges to $\mathrm{H}_*(\mathrm{hoTot}(\mathcal{K}_d^\bullet))$ when $d \geq 4$ [31, Theorem 7.2]. Our main result is

Theorem 1.2. *For $d \geq 4$ the homology Bousfield-Kan spectral sequence associated to Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ collapses at the E^2 page rationally.*

Before looking at the consequences of this theorem, we give an overview of its proof. The key idea is a relative version of Kontsevich's theorem on the formality of the little d-disks operad [18, Section 3], which states that there is a chain of quasi-isomorphisms of operads between the singular chains of the little d-disks operad and its homology with real coefficients. This is therefore true for any operad weakly equivalent to the little d-disks operad. In particular there is a chain of quasi-isomorphisms $\mathrm{C}_*(\mathcal{K}_d; \mathbb{R}) \simeq \mathrm{H}_*(\mathcal{K}_d; \mathbb{R})$. If one could deduce from this the formality of the cosimplicial space $\mathcal{K}_d^\bullet$, the collapse of the homology Bousfield-Kan spectral sequence would immediately follow from Proposition 3.2. For this it would be enough to know, by McClure-Smith, that $\mathcal{K}_d$ is formal as a *multiplicative* operad. But this does not appear to be easy to establish (see Remark 7.4), and we do not know whether $\mathcal{K}_d^\bullet$ is formal.

Paper sample

This is why we make a detour through the *Fulton-MacPherson operad* $\mathcal{F}_d$. This operad is homotopy equivalent to $\mathcal{K}_d$ but is more suitable for proving formality results as those in [18]. It is not multiplicative, but it is "multiplicative up to homotopy" in the sense that it admits a map from the contractible A_∞ Stasheff associahedral operad. The key formality statement that we prove is that when $d \geq 3$, $\mathcal{F}_d$ is formal as an operad which is "multiplicative up to homotopy" (Lemma 7.3). However, the problem is that we cannot directly associate to $\mathcal{F}_d$ a cosimplicial space in the spirit of McClure-Smith. We instead construct certain finite diagrams $\widehat{\mathcal{F}}_{d\{n\}}$ of spaces which we call *fanic diagrams*. These are built out of the up-to-homotopy multiplicative structure on $\mathcal{F}_d$ and are in a sense a rigid version of a "cosimplicial space up to homotopy" analogous to $\mathcal{K}_d^\bullet$, or more precisely analogous to its nth truncation. Our formality statement implies that $\widehat{\mathcal{F}}_{d\{n\}}$ is formal (Theorem 7.1) and hence the homology Bousfield-Kan spectral sequence of the cosimplicial replacement of that diagram collapses at the E^2 page. Thus the same must be true for the cosimplicial replacement of the nth truncation $\mathcal{K}_{d[n]}$ of $\mathcal{K}_d^\bullet$. Using the fact that, for each $n \geq 0$, the homology spectral sequence of $\mathcal{K}_{d[n]}$ collapses, along with a strong convergence result for Bousfield-Kan spectral sequences as it applies to $\mathcal{K}_d^\bullet$, we deduce Theorem 1.2.

We now come back to the Vassiliev spectral sequence and explain its link with the Bousfield-Kan spectral sequence. In his thesis, the second author found a more conceptual description of the E^1 page of the Vassiliev spectral sequence than the original combinatorial one arising from the classification of singularities in the discriminant set. To explain this, recall that the *Poisson operad with bracket of degree* $d-1$, $\mathcal{POISS}_{d-1}$, is the operad encoding Poisson algebras, i.e. graded commutative algebras equipped with a Lie bracket of degree $(d-1)$ which is a graded derivation with respect to the multiplication (see, for instance, Example (d) in [33, Section 1]). The work of Fred Cohen [8] implies that, for $d \geq 2$, $\mathcal{POISS}_{d-1}$ is just the homology of the topological little d-disks operad. Since Poisson algebras admit an associative multiplication $m \in \mathcal{POISS}_{d-1}(2)$, we can define a differential

$$\delta\colon \mathcal{POISS}_{d-1}(n) \longrightarrow \mathcal{POISS}_{d-1}(n+1)$$

where $\delta = [m, -]$ is the Gerstenhaber bracket and $\delta^2 = 0$ by associativity [33, Section 3]. The homology of the cochain complex $(\mathcal{POISS}_{d-1}(*), \delta)$ is called the *Hochschild homology* of that multiplicative operad, denoted by $\mathrm{HH}_*(\mathcal{POISS}_{d-1})$. This complex is in fact the deformation complex of the morphism of operads $\mathcal{ASS} \to \mathcal{POISS}_{d-1}$, where $\mathcal{ASS}$ is the associative algebras operad [19]. Finally, a slight variation of Vassiliev's spectral sequence for $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ produces a spectral sequence for $\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d)$ and we have

Theorem 1.3 (Turchin). *The E^1 page of the Vassiliev spectral sequence computing $\mathrm{H}_*(\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d))$ is isomorphic to the Hochschild homology of the Poisson operad with bracket of degree $d-1$.*

This statement was first established for the Vassiliev spectral sequence for $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ [33], and the above analogous result is a combination of [35, Theorem 8.4] and [34, Proposition 3.1 and Lemma 4.3].

On the other hand, the E^2 page of the Bousfield-Kan spectral sequence for $\mathcal{K}_d^\bullet$ is by definition the homology of the conormalisation of the cosimplicial abelian group $\mathrm{H}_*(\mathcal{K}_d^\bullet)$. Since the cofaces are induced by the multiplicative structure on $\mathcal{K}_d$ and since the homology of that multiplicative topological operad is exactly the multiplicative Poisson operad, it is easy to deduce that this E^2 page is also isomorphic to $\mathrm{HH}_*(\mathcal{POISS}_{d-1})$ [31, Corollary 1.3]. This implies that the E^1 page of the Vassiliev spectral sequence for $\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d)$ and the E^2 page of the homology spectral sequence for $\mathcal{K}_d^\bullet$ are isomorphic. It can be shown that this isomorphism is just a regrading [35, Proposition 0.1]. Since both of these spectral sequences converge to $\mathrm{H}_*(\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d); \mathbb{Q})$ for $d \geq 4$, Theorem 1.2

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implies that the Vassiliev spectral sequence collapses at E^1. As an immediate consequence of this and (1.1) we have the following

Corollary 1.4. *For* $d \geq 4$,

$$\mathrm{H}_*(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d); \mathbb{Q}) \otimes \mathrm{H}_*(\Omega^2 S^{d-1}; \mathbb{Q}) \cong \mathrm{H}_*(\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d); \mathbb{Q}) \cong \mathrm{HH}_*(\mathcal{POISS}_{d-1}).$$

The rational homology of $\Omega^2 S^{d-1}$ is isomorphic to a free graded commutative algebra on one or two generators depending on the parity of d, so it is very simple. In addition, the E^2 page of the Bousfield-Kan spectral sequence for $\mathcal{K}_d^\bullet$, or equivalently the Hochschild homology of the Poisson operad, can be expressed as the homology of an explicit cochain complex described in terms of the homology of configuration spaces in $\mathbb{R}^d$ which are well understood. In particular, there is an algorithm for computing this homology which has been used in [35]. Thus, in theory, $\mathrm{H}_*(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d); \mathbb{Q})$ is completely computable. However, it appears that the algorithmic complexity is exponential and the only feasible computations are all still in low degrees. What is necessary, and is still lacking, is a deeper understanding of the structure of $\mathrm{HH}_*(\mathcal{POISS}_{d-1})$.

It is easy to see that the homology of the cosimplicial space $\mathcal{K}_d^\bullet$ depends, up to an obvious regrading, only on the parity of d. This implies that, up to regrading, all the E^2 pages of the homology Bousfield-Kan spectral sequences of $\mathcal{K}_d^\bullet$ for d even (respectively d odd) are isomorphic. This completes the proof of the characterization of the rational homology of spaces of long knots as stated in [17, Theorem 2.3]. Furthermore, the operation of stacking of long knots gives a multiplication which makes $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ into an H-space. Therefore the rational homotopy type of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ for $d \geq 4$ is completely determined by its rational homology and is thus virtually known by our main theorem.

Coming back to the original Vassiliev conjecture, notice that by [34, Theorem 5.1] and results of [37, Section IV], E^1 page of the Vassiliev spectral sequence for $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ and the corresponding $\overline{E^1}$ page of the Vassiliev spectral sequence for $\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d)$ are related by an isomorphism $E^1 \otimes \mathrm{H}_*(\Omega^2 S^{d-1}) \cong \overline{E^1}$. Therefore Corollary 1.4 and the isomorphism $\overline{E^1} \cong \mathrm{HH}_*(\mathcal{POISS}_{d-1})$ from Theorem 1.3 imply the collapse at E^1 of the classical Vassiliev spectral sequence for $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$, which is our promised Theorem 1.1.

The *homotopy* Bousfield-Kan spectral sequence computing the homotopy groups of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$ from the cosimplicial abelian group $\pi_*(\mathcal{K}_d^\bullet)$ has been studied elsewhere [1]. Using the remarkable fact that the little d-disks operad is not only formal (i.e. rationally determined by its homology), but also *coformal* (i.e. rationally determined by its homotopy), it has been shown in [1] that Theorem 1.2 implies that the homotopy spectral sequence tensored with the rationals also collapses at the E^2 page. A strong connection between the E^2 pages of the rational homology and the homotopy spectral sequences is exhibited in [20] where it is also shown that the homotopy groups of the space of long knots are also the homology of a certain graph complex, smaller than the one used for computing the homology of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^d)$. Some computations in this E^2 page appeared in [28].

A natural next question is whether our approach gives the collapse of the Vassiliev spectral sequence for long knots in $\mathbb{R}^3$. There are many difficult issues here, not the least of which is that it is not even clear what the Vassiliev spectral sequence converges to. However, we are nevertheless able to prove a certain collapsing result. Indeed, Proposition 8.1 states that the homology Bousfield-Kan spectral sequence of the cosimplicial replacement of the nth truncation $\mathcal{K}_{d[n]}$ collapses at E^2. This is true for $d \geq 3$, but to deduce the collapse of the Bousfield-Kan spectral sequence for $\mathcal{K}_d^\bullet$ itself we need some convergence hypotheses which are only true for $d \geq 4$. It at least seems plausible that this spectral sequence collapses for $d = 3$. Another interesting question is whether our result can be extended over the integers.

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Finally, it is possible to extend our results to spaces of embeddings of more general manifolds than $\mathbb{R}$ in $\mathbb{R}^d$. This is done by generalizing the following slogan which summarizes our proof of Theorem 1.2:

Goodwillie-Weiss embedding calculus + Kontsevich formality of the little d-disks operad	$\Longrightarrow$	collapse of spectral sequences for spaces of embeddings

This slogan was taken much further in [2]. The authors of that paper use Weiss' orthogonal calculus to prove a collapse result for a spectral sequence computing the rational homology of $\overline{\mathrm{Emb}}(M, \mathbb{R}^d)$ for any compact manifold M and d large enough. In particular, it is shown that for large d, the rational homology of the space of embeddings modulo immersions, $\mathrm{H}_*(\overline{\mathrm{Emb}}(M, \mathbb{R}^d); \mathbb{Q})$, depends only on the rational homotopy type of M.

1.1. **Organization of the paper.** In Section 2 we review basic facts on cosimplicial spaces and diagrams. We also recall Sinha's cosimplicial model for the space of long knots, the notion of a multiplicative operad and the associated Gerstenhaber-Voronov/McClure-Smith cosimplicial object, and the definitions of the Kontsevich and Fulton-MacPherson operads. Formality, which comes from rational homotopy theory, is recalled in Section 3 where we also prove that the homology Bousfield-Kan spectral sequence of a formal cosimplicial space, or more generally of the cosimplicial replacement of a formal finite diagram, collapses at the E^2 page. In Section 4 we study a condition on a cosimplicial space that guarantees that the Bousfield-Kan spectral sequences for its truncations converge. In Section 5 we introduce the category of *fans*, which is a variation of the truncated cosimplicial category $\Delta[n]$, and in Section 6 show how to associate to any morphism of nonsymmetric operads a diagram shaped on the category of fans. This generalizes the Gerstenhaber-Voronov/McClure-Smith cosimplicial diagram associated to a multiplicative operad. In Section 7 we prove a relative version of Kontsevich's formality of the little d-disks operad and deduce the formality of the fanic diagrams associated to the Kontsevich multiplicative operad. In the last section we collect these results to give a proof of our main theorem.

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2. The Kontsevich and Fulton-MacPherson operads and Sinha's cosimplicial model for the space of long knots

In this section we review Sinha's cosimplicial model for the space of long knots and its relation to the Kontsevich and Fulton-MacPherson operads. To start, we review some standard facts about cosimplicial spaces and diagrams as well as the construction of a cosimplicial space associated to a multiplicative operad.

2.1. **Cosimplicial spaces, homotopy totalizations, Bousfield-Kan spectral sequences, cosimplicial replacement of diagrams, and left cofinal functors.** The standard references for the following basic terminology and facts about cosimplicial objects are [6, X] and [40, Chapter 8].

The simplicial category Δ has ordered sets $[n] := \{0, 1, 2, ..., n\}$, $n \geq 0$, as objects and order-preserving maps as morphisms. All morphisms in Δ are compositions of *cofaces* $d^i\colon [n] \to [n+1]$ and *codegeneracies* $s^j\colon [n] \to [n-1]$. A *cosimplicial object* in a category $\mathcal{C}$ is a covariant functor

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from Δ to $\mathcal{C}$. Dually, a *simplicial object* in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. In particular a *(pointed) cosimplicial space* is a covariant functor

$$X^\bullet \colon \Delta \longrightarrow \mathrm{Top} \quad (\text{or } \mathrm{Top}_*)$$

from the simplicial category to the category of (pointed) spaces. The *standard cosimplicial space* is the cosimplicial space $\Delta^\bullet$ where Δ^n is the standard geometric n-simplex, the cofaces are defined from the inclusions of faces $\Delta^n \hookrightarrow \Delta^{n+1}$, and the codegeneracies are suitable affine projections $\Delta^n \to \Delta^{n-1}$.

The *totalization* of a cosimplicial space $X^\bullet$, denoted by $\mathrm{Tot}\, X^\bullet$, is the space of natural maps from the standard cosimplicial space $\Delta^\bullet$ to $X^\bullet$,

$$\mathrm{Tot}\, X^\bullet := \mathrm{Hom}_\Delta(\Delta^\bullet, X^\bullet).$$

When the cosimplicial space $X^\bullet$ is *fibrant* [6, X.4.6], $\mathrm{Tot}(X^\bullet)$ is homotopy equivalent to the homotopy limit $\mathrm{holim}_\Delta X^\bullet$ [6, XI.4.4]. As explained in [5, Section 2.7], the techniques of [6, pp.279-280] imply that any cosimplicial space $X^\bullet$ admits a weakly equivalent fibrant functorial replacement $\tilde{X}^\bullet$. One then defines the *homotopy totalization* of $X^\bullet$ by

$$\mathrm{hoTot}\, X^\bullet := \mathrm{Tot}\, \tilde{X}^\bullet.$$

Since by definition $\tilde{X}^\bullet$ is fibrant and weakly equivalent to $X^\bullet$, $\mathrm{hoTot}\, X^\bullet$ is always weakly equivalent to $\mathrm{holim}_\Delta X^\bullet$. This homotopy totalization hoTot is also weakly equivalent to the homotopy invariant totalization $\widetilde{\mathrm{Tot}}$ used in [31].

Consider the full subcategory $\Delta[n] \subset \Delta$ consisting of objects $[0], \ldots, [n]$. The *nth truncation* of a cosimplicial object $X^\bullet$ in $\mathcal{C}$ is the composite

$$X_{[n]} : \Delta[n] \hookrightarrow \Delta \xrightarrow{X^\bullet} \mathcal{C}.$$

The *nth partial homotopy totalization* of $X^\bullet$ is defined as

$$\mathrm{hoTot}^n(X^\bullet) := \operatorname*{holim}_{\Delta[n]} X_{[n]}.$$

To any cosimplicial space $X^\bullet$ one can associate a second quadrant homology Bousfield-Kan spectral sequence with cofficients in an abelian group A [5], and a homotopy Bousfield-Kan spectral sequence if $X^\bullet$ is pointed [6]. These converge under favorable circumstances to $\mathrm{H}_*(\mathrm{hoTot}\, X^\bullet; A)$ or $\pi_*(\mathrm{hoTot}\, X^\bullet)$. We state certain strong convergence conditions in Section 4.

A *diagram of spaces* is a covariant functor $F \colon I \to \mathrm{Top}$ where I is a small category called the *shape* of the diagram. We will also use the notion of a *cosimplicial replacement of the diagram* F as defined in [6, XI.5], denoted by $\Pi^\bullet F$. The nth term of this cosimplicial space is given by

$$\Pi^n F := \prod_{u=(i_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} i_n) \in \mathrm{N}_n(I)} F(i_0)$$

where the simplicial set $\mathrm{N}_\bullet(I)$ is the nerve of the category I. The cofaces and codegeneracies are obtained as suitable diagonal maps and projections. By [6, XI.5.2], the (homotopy) totalization of this cosimplicial space is weakly equivalent to the homotopy limit of F, i.e.

$$\mathrm{hoTot}(\Pi^\bullet F) \simeq \operatorname*{holim}_I F.$$

If $\theta \colon J \to I$ is a functor then the I-diagram F induces a J-diagram $\theta^* F := F \circ \theta$. This *change of shapes* functor θ is said to be *left cofinal* if for every object i in I, the overcategory $\theta \downarrow i$ is contractible (see [6, XI.9] with $\theta \downarrow i$ defined and denoted by θ/i in [6, XI.2.2]). In this case there is a weak equivalence

$$\operatorname*{holim}_I F \simeq \operatorname*{holim}_J \theta^* F.$$

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2.2. **The construction of a cosimplicial object associated to a multiplicative operad following McClure and Smith.** Here we recall the notion of the cosimplicial object associated to a multiplicative operad from [26, Section 3].

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category where the object $\mathbf{1}$ is the unit for $\otimes$. A *nonsymmetric operad* $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ is a collection of objects of $\mathcal{C}$ with all the properties of an operad except those having to do with the actions of the symmetric groups [25, Definition 3.12]. Notice that as part of the definition there is a unit morphism $\mathrm{id}\colon \mathbf{1} \to \mathcal{O}(1)$ and we also suppose that a (nonsymmetric) operad has an object in degree 0, contrary to the definition in [24]. We denote by

$$\circ_i\colon \mathcal{O}(p) \otimes \mathcal{O}(q) \longrightarrow \mathcal{O}(p+q-1), \quad 1 \leq i \leq p,$$

the usual insertion operations induced by the operad structure as defined for example in [24, p. 7]. The *associative nonsymmetric operad* $\mathcal{ASS}$ is defined by $\mathcal{ASS}(n) = \mathbf{1}$ for each $n \geq 0$, with operadic structure maps the standard isomorphisms $\mathbf{1}^{\otimes(1+n)} \cong \mathbf{1}$.

A *multiplicative operad* is a nonsymmetric operad $\mathcal{O}$ equipped with a morphism of nonsymmetric operads $\mu\colon \mathcal{ASS} \to \mathcal{O}$, ([12],[26, Definition 3.1 and Remark 3.2 (i)]). Such a multiplicative structure on $\mathcal{O}$ is equivalent to having morphisms $e\colon \mathbf{1} \to \mathcal{O}(0)$ and $m\colon \mathbf{1} \to \mathcal{O}(2)$ satisfying

$$m \circ_1 m = m \circ_2 m \quad \text{and} \quad m \circ_1 e = m \circ_2 e = \mathrm{id}. \tag{2.1}$$

One can associate a cosimplicial object $\mathcal{O}^\bullet$ to any multiplicative operad $\mathcal{O}$ [26, Section 3] by defining $\mathcal{O}^n = \mathcal{O}(n)$ with the cofaces and codegeneracies given by the following formulas. For $x \in \mathcal{O}(n)$,

$$\begin{aligned} d^0(x) &= m \circ_2 x, \\ d^i(x) &= x \circ_i m, \quad \text{for } 1 \leq i \leq n, \\ d^{n+1}(x) &= m \circ_1 x, \\ s^j(x) &= x \circ_j e, \quad \text{for } 1 \leq j \leq n. \end{aligned}$$

It is easy to check that the cosimplicial identities are consequences of Equation (2.1).

There are also obvious dual notions of a *cooperad*, of a *coassociative cooperad* (also denoted by $\mathcal{ASS}$), of a *comultiplicative cooperad*, and of a simplicial object associated to a comultiplicative cooperad.

2.3. **The Kontsevich multiplicative operad and Sinha's cosimplicial model.** Fix $d \geq 1$ and recall that a linear embedding $\epsilon\colon \mathbb{R} \hookrightarrow \mathbb{R}^d$ has also been fixed.

The Kontsevich operad $\mathcal{K}_d = \{\mathcal{K}_d(n)\}_{n\geq 0}$ is defined and studied in [31, Definition 4.1 and Theorem 4.5]. Each space $\mathcal{K}_d(n)$ is obtained as a suitable compactification of the ordered configuration space of n points in $\mathbb{R}^d$ modulo the action of $\mathbb{R} \ltimes \mathbb{R}^d$ by scaling and translation. Another feature is that colinear configurations are identified. More precisely, $\mathcal{K}_d(n)$ is the closure of the image of the map

$$\alpha_* = (\alpha_{ij})_{1\leq i<j\leq n}\colon C(n, \mathbb{R}^d) \longrightarrow \prod_{1\leq i<j\leq n} S^{d-1} \tag{2.2}$$

where $C(n, \mathbb{R}^d)$ is the space of configurations of n points in $\mathbb{R}^d$ and $\alpha_{ij}\colon C(n, \mathbb{R}^d) \to S^{d-1}$ is defined by $\alpha_{ij}(x_1, \ldots, x_n) = (x_i - x_j)/\|x_i - x_j\|$. It can be shown that this operad is homotopy equivalent to the classical little d-disks operad.

The spaces $\mathcal{K}_1(n)$ turn out to be homeomorphic to the discrete symmetric group Σ_n on n letters, since all configuration on $\mathbb{R}$ are colinear. Let $\mathcal{K}_1^{(0)}(n)$ be the path-connected component of $\mathcal{K}_1(n)$ corresponding to the linearly ordered configuration $(1, \ldots, n)$ on the line. Since $\mathcal{K}_1^{(0)}(n)$ is a one-point space, it is clear that $\mathcal{K}_1^{(0)} = \mathcal{ASS}$, the associative nonsymmetric operad in the monoidal cartesian category of spaces.

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The linear embedding ϵ induces a morphism of operads

$$\epsilon_{\#} \colon \mathcal{K}_1 \longrightarrow \mathcal{K}_d$$

which sends a configuration on the line to its image under ϵ in $\mathbb{R}^d$. This restricts to a morphism

$$\epsilon_{\#} \colon \mathcal{K}_1^{(0)} = \mathcal{ASS} \longrightarrow \mathcal{K}_d$$

which endows $\mathcal{K}_d$ with the structure of a multiplicative operad.

One can thus associate a cosimplicial space $\mathcal{K}_d^{\bullet}$, which we will call *Sinha's cosimplicial space*, to $\mathcal{K}_d$, as outlined in Section 2.2. In more detail, this is a cosimplicial space

$$\mathcal{K}_d^{\bullet} = (\mathcal{K}_d(0) \rightrightarrows \mathcal{K}_d(1) \rightrightarrows \mathcal{K}_d(2) \cdots),$$

where $\mathcal{K}_d(n)$ has the homotopy type of the space of configurations of n points in $\mathbb{R}^d$. Cofaces d^i corresponds to "doubling" the ith point of the configuration "infinitesimally" in the direction given by ϵ and codegeneracies s^i forget the ith point in the configuration. This is explained in detail in [29, 31].

We have the following important result due to Sinha.

Theorem 2.1 ([31], Corollary 1.2)**.** *For $d \geq 4$, the space of long knots modulo immersion is weakly equivalent to the homotopy totalization of the cosimplicial space associated to the Kontsevich operad, i.e.*

$$\overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{hoTot}(\mathcal{K}_d^{\bullet}).$$

2.4. **The Fulton-MacPherson operad.** We recall here the Fulton-MacPherson operad and its relation to the Kontsevich operad. We will need this operad to establish certain formality results. Our main reference is [30], although this operad is also studied in [3, 11, 13, 18, 23].

The Fulton-MacPherson operad, $\mathcal{F}_d = \{\mathcal{F}_d(n)\}_{n \geq 0}$, is a topological operad whose nth term is also a compactification of $C(n, \mathbb{R}^d)$ modulo scaling and translation. The difference between this and the compactification defining the Kontevich operad is that no identification of the collinear configurations takes place. More precisely, $\mathcal{F}_d(n)$ is defined as the closure of the image of the map

$$(\alpha_*, \beta_*) \colon C(n, \mathbb{R}^d) \to \prod_{1 \leq i < j \leq n} S^{d-1} \times \prod_{1 \leq i < j < k \leq n} [0, \infty] \tag{2.3}$$

where $\alpha_* = (\alpha_{ij})_{1 \leq i < j \leq n}$ is as in (2.2) and $\beta_* = (\beta_{ijk})_{1 \leq i < j < k \leq n}$ is defined by $\beta_{ijk}(x_1, \cdots, x_n) = \|x_i - x_j\| / \|x_i - x_k\|$ [30, Definition 4.11]. There is a morphism of operads

$$q \colon \mathcal{F}_d \longrightarrow \mathcal{K}_d \tag{2.4}$$

induced by the obvious projection between the target spaces of maps (2.3) and (2.2). Each $q \colon \mathcal{F}_d(n) \to \mathcal{K}_d(n)$ is a homotopy equivalence [30, Corollary 5.9] (see also [31, Theorem 4.2]), as is the map of operads (2.4) [27] (see also [22, Section 2]).

Denote by $\mathcal{F}_1^{(0)}(n)$ the path component in $\mathcal{F}_1(n)$ containing the linearly ordered configuration $(1, \ldots, n)$ on the line. This defines a nonsymmetric operad $\mathcal{F}_1^{(0)}$ which is homeomorphic to the Stasheff operad. In particular $\mathcal{F}_1^{(0)}(n)$ is the nth associahedron which is a convex polytope of dimension $n-2$ (or 0 for $n < 2$) [30, Section 4.4]. The linear embedding ϵ induces a morphism $\epsilon_{\#} \colon \mathcal{F}_1^{(0)} \to \mathcal{F}_d$ and we have a commutative diagram of nonsymmetric operads

$$\begin{array}{ccc} \mathcal{F}_1^{(0)} & \xrightarrow{\epsilon_{\#}} & \mathcal{F}_d \\ {\scriptstyle q}\downarrow{\scriptstyle \simeq} & & {\scriptstyle q}\downarrow{\scriptstyle \simeq} \\ \mathcal{ASS} = \mathcal{K}_1^{(0)} & \xrightarrow{\epsilon_{\#}} & \mathcal{K}_d \end{array} \tag{2.5}$$

where the vertical maps are homotopy equivalences.

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The operad $\mathcal{F}_d$ is not multiplicative but since each $\mathcal{F}_1^{(0)}(n)$ is contractible, $\epsilon_\#$ in a sense endows $\mathcal{F}_d$ with a multiplicative structure "up to homotopy". Thus we cannot apply the Gerstenhaber-Voronov/McClure-Smith construction directly to obtain a cosimplicial space out of $\mathcal{F}_d$, but we will see in Section 6 that one can associate a more general diagram to $\mathcal{F}_d$ which generalizes this construction.

3. Formality and collapse of the homology Bousfield-Kan spectral sequence

In this section we recall the notion of a formal diagram in rational homotopy theory and we show that the homology Bousfield-Kan spectral sequence of a formal cosimplicial space collapses at the E^2 page. We also show that formality of a finite diagram is preserved by passing to its cosimplicial replacement.

We first review some classical notions in rational homotopy theory for which [10] is the standard reference. Let $\mathbb{K}$ be a field of characteristic 0 and denote by CDGA the category of commutative differential graded algebras. Let

$$A_{PL}(-;\mathbb{K})\colon \mathrm{Top} \to \mathrm{CDGA}$$

be Sullivan's functor of piecewise polynomial forms as defined in [10, §10 (c)] and recall that, for a space X, $H(A_{PL}(X;\mathbb{K})) \cong H^*(X;\mathbb{K})$. In fact $A_{PL}(-;\mathbb{K})$ is naturally connected by a zig-zag of quasi-isomorphisms of cochain complexes to the singular cochains $\mathrm{C}^*(-;\mathbb{K})$ [10, §10(e)]. The definition of a *formal space* was introduced in [9], and it can be generalized to diagrams as follows.

Definition 3.1. Let I be a small category and let $\mathbb{K}$ be a field of characteristic 0. A functor $A\colon I \to \mathrm{CDGA}$ is called *formal* if it is connected by a zig-zag of natural quasi-isomorphism to its homology $\mathrm{H}(A)$. A diagram of spaces $F\colon I \to \mathrm{Top}$ is called $\mathbb{K}$-*formal* if the contravariant functor $A_{PL}(F;\mathbb{K})\colon I \to \mathrm{CDGA}$ is formal.

As a special case, a cosimplicial space $X^\bullet$ is formal if the diagram $X^\bullet\colon \Delta \to \mathrm{Top}$ is formal. Our main interest in this is the following collapsing result.

Proposition 3.2. *Let $X^\bullet$ be a cosimplicial space and let $\mathbb{K}$ be a field of characteristic 0. If the cosimplicial space $X^\bullet$ is $\mathbb{K}$-formal then the homology Bousfield-Kan spectral sequence for $X^\bullet$ with coefficients in any field of characteristic 0 collapses at the E^2 page.*

Proof. We consider the homology Bousfield-Kan spectral sequence as constructed in [5, Section 2.1]. For a cosimplicial chain complex $V_*^\bullet$, denote by $\{E^r(V_*^\bullet)\}_{r\geq 0}$ the spectral sequence induced by the filtration by cosimplicial degree in the associated total complex of the bicomplex $\prod_{m\geq 0} N^m(V_*^\bullet)$. Here N^* is the conormalization as defined in [5, Section 2] or in [40, Chapter 8]. It is clear that $\{E^r(C_*(X^\bullet;\mathbb{K}))\}_{r\geq 0}$ coincides from the E^2 page with the homology Bousfield-Kan spectral sequence of $X^\bullet$. The $\mathbb{K}$-formality of the cosimplicial space and the natural equivalence between A_{PL} and singular cochains imply that there is a zig-zag of quasi-isomorphisms

$$C_*(X^\bullet;\mathbb{K}) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} H_*(X^\bullet;\mathbb{K}).$$

Therefore the spectral sequence $\{E^r(C_*(X^\bullet;\mathbb{K}))\}_{r\geq 0}$ coincides from the E^1 page with $\{E^r(H_*(X^\bullet;\mathbb{K}))\}_{r\geq 0}$. But the latter spectral sequence collapses at E^2 because each $H_*(X^n;\mathbb{K})$ is a chain complex with 0 differential so that the vertical differential in the associated bicomplex is trivial. This proves the statement for $\mathbb{K}$. We also have an isomorphism of spectral sequences

$$\{E^r(C_*(X^\bullet;\mathbb{K}))\}_{r\geq 0} \cong \{E^r(C_*(X^\bullet;\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{K}\}_{r\geq 0},$$

and therefore the spectral sequence with rational coefficients collapses at the same page. Applying this argument in the opposite direction proves the result for any field of characteristic 0. $\square$

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If we could prove that Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ was formal we would deduce immediately from the last proposition the collapse of the associated homology Bousfield-Kan spectral sequence as claimed in Theorem 1.2. Unfortunately, it seems difficult to prove formality of $\mathcal{K}_d^\bullet$ directly, so we will prove formality of some other diagrams approximating $\mathcal{K}_d^\bullet$ instead. But in order to deduce collapsing results from formality of these auxiliary diagrams, we now need to show that the cosimplicial replacement of a formal finite diagram is a formal cosimplicial space.

A category I is said to be *finite* if it has a finite number of morphisms (hence of objects).

Proposition 3.3. *Let I be a finite category, let $\mathbb{K}$ be a field of characteristic 0, and let $F\colon I \to \mathrm{Top}$ be a diagram. If F is $\mathbb{K}$-formal then so is its cosimplicial replacement $\Pi^\bullet F$.*

Proof. Recall from [6, XI.5] and from the end of Section 2.1 the definition of the cosimplicial replacement $\Pi^\bullet F$. Inspired by this construction we associate to a *contravariant* functor $A\colon I \to$ CDGA a simplicial CDGA, which we denote by $\otimes_\bullet A$ and define as follows. Let $\mathrm{N}_\bullet(I)$ be the nerve of I and denote by

$$u = \left(i_0 \overset{f_1}{\leftarrow} \cdots \overset{f_n}{\leftarrow} i_n\right)$$

a typical element of $\mathrm{N}_n(I)$. The nth term of $\otimes_\bullet A$ is given by

$$(\otimes_\bullet A)_n := \bigotimes_{u\in \mathrm{N}_n(I)} A(i_0).$$

Before defining the cofaces, notice that for $u \in \mathrm{N}_n(I)$, taking all possible factorizations $f_i = f'' \circ f'$ for a fixed $1 \leq i \leq n$, postcomposing by all possible maps $g\colon i_{-1} \leftarrow i_0$ with source i_0, or precomposing with all possible maps $h\colon i_n \leftarrow i_{n+1}$ with target i_n gives isomorphisms

$$(\otimes_\bullet A)_{n+1} \cong \begin{cases} \otimes_{u\in \mathrm{N}_n(I)} \otimes_{f_i = f'\circ f''} A(i_0) \\ \otimes_{u\in \mathrm{N}_n(I)} \otimes_{g\colon i_{-1}\leftarrow i_0} A(i_{-1}) \\ \otimes_{u\in \mathrm{N}_n(I)} \otimes_{h\colon i_n\leftarrow i_{n+1}} A(i_0). \end{cases}$$

The faces ∂_i are defined using these isomorphisms and the multiplicative structure:

$$\partial_0 = \bigotimes_{u\in \mathrm{N}_n(I)} \left(\bigotimes_{g\colon i_{-1}\leftarrow i_0} A(i_{-1}) \xrightarrow{\Psi_u} A(i_0) \right)$$

$$\partial_i = \bigotimes_{u\in \mathrm{N}_n(I)} \left(\bigotimes_{f_i = f'\circ f''} A(i_0) \xrightarrow{\text{mult}} A(i_0) \right), \quad 1 \leq i \leq n,$$

$$\partial_{n+1} = \bigotimes_{u\in \mathrm{N}_n(I)} \left(\bigotimes_{h\colon i_n\leftarrow i_{n+1}} A(i_0) \xrightarrow{\text{mult}} A(i_0) \right),$$

where $\Psi_u(\otimes_g x_g)$ is the product of the $A(g)(x_g)$ over the finite set of maps $g\colon i_{-1} \leftarrow i_0$ with source i_0, for $x_g \in A(i_{-1})$. To define the codegeneracies σ_j we use the unit map $\eta\colon \mathbb{K} \to A(i)$ associated to any CDGA and we set

$$\sigma_j\colon \bigotimes_{u'\in \mathrm{N}_{n-1}(I)} A(i_0') \cong \bigotimes_{u\in \mathrm{N}_n(I), f_j=\mathrm{id}} A(i_0) \xrightarrow{\mathrm{id}\otimes\eta} \bigotimes_{u\in \mathrm{N}_n(I)} A(i_0).$$

We call $\otimes_\bullet A$ the *simplicial replacement* of the contravariant functor A. Notice that we need the nerve of I to be finite in each degree for the above maps to be well defined.

When $F\colon I \to \mathrm{Top}$ is a formal diagram of spaces, applying this simplicial replacement to each term in the zigzag of CDGA quasi-isomorphisms connecting $A_{PL}(F)$ and $\mathrm{H}^*(F)$ gives a zig-zag

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of quasi-isomorphisms of simplicial CDGAs

$$\otimes_\bullet A_{PL}(F) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \otimes_\bullet \mathrm{H}^*(F). \tag{3.1}$$

Kunneth (quasi-)isomorphisms induce a quasi-isomorphism of simplicial CDGAs

$$\otimes_\bullet A_{PL}(F) \xrightarrow{\simeq} A_{PL}(\Pi^\bullet F),$$

and similarly the standard Kunneth isomorphism gives an isomorphism

$$\otimes_\bullet \mathrm{H}^*(F) \xrightarrow{\cong} \mathrm{H}^*(\Pi^\bullet F).$$

Combining this with the quasi-isomorphisms (3.1) proves the formality of $\Pi^\bullet F$. □

4. Convergence of Bousfield-Kan spectral sequences for cosimplicial spaces and their truncations

Throughout this section, $X^\bullet$ is a simply-connected pointed cosimplicial space. We will establish a condition on such a cosimplicial space which guarantees that the Bousfield-Kan spectral sequences of its truncations converge. We also prove a folklore theorem which says that a left cofinal change of shape preserves the E^2 pages of the homotopy and homology Bousfield-Kan spectral sequences.

Our main convergence hypotheses will be the following

Definition 4.1. We say that $X^\bullet$ is *well above the diagonal at page* E^r if that page of the second-quadrant homotopy Bousfield-Kan spectral sequence associated to $X^\bullet$ satisfies the conditions

(i) $E^r_{-p,q} = 0$ for $q \leq p$, and
(ii) for each i there are finitely many p such that $E^r_{-p,p+i} \neq 0$.

In other words, these conditions say that there are no terms on the main diagonal $E^r_{-p,p}$ or below, and that there are finitely many terms on each diagonal line of slope -1. Such conditions already appear in [5].

Consider now the nth truncation $X_{[n]}$ of $X^\bullet$. To prove a convergence result for the Bousfield-Kan spectral sequences of its cosimplicial replacement $\Pi^\bullet(X_{[n]})$, we need the following lemma relating the homotopy spectral sequences associated to $X^\bullet$ and $\Pi^\bullet(X_{[n]})$.

Recall the conormalisation functor $\mathrm{N}^*\colon \mathrm{Ab}^\Delta \to \mathrm{Ch}^{\geq 0}$ from cosimplicial abelian groups to non-negatively graded cochain complexes as defined in [40, Section 8.4] or in [5, Section 2]. For a cochain complex C^* define its *nth truncation* $\tau^n C^*$ to be the cochain complex given by

$$\begin{cases} (\tau^n C^*)^q &= C^q \text{ if } q \leq n \\ (\tau^n C^*)^q &= 0 \text{ if } q > n \end{cases}$$

with the differential induced in the obvious way from the differential on C^*.

Lemma 4.2. *The E^2 page of the homotopy Bousfield-Kan spectral sequence associated to $\Pi^\bullet(X_{[n]})$ is given by*

$$E^2_{-p,q} = \mathrm{H}^p(\tau^n \mathrm{N}^*(\pi_q(X^\bullet))).$$

Proof. The following proof was given to us by P. Bousfield. By [6, XI.7.1] we have an isomorphism $E^2_{-p,q} \cong \lim^p_{\Delta[n]} \pi_q(X_{[n]})$, where $\lim^p$ is the pth left derived functor of $\lim^0 = \lim$. Recall for example from [40, 8.4] the Dold-Kan Theorem which states that the conormalisation functor $\mathrm{N}^*\colon \mathrm{Ab}^\Delta \to \mathrm{Ch}^{\geq 0}$ is an equivalence of abelian categories. It is not difficult to adapt the proof of that result to get an equivalence of abelian categories

$$\mathrm{N}^*_n\colon \mathrm{Ab}^{\Delta[n]} \longrightarrow \mathrm{Ch}^{[n]}$$

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between n-truncated cosimplicial abelian groups and cochain complexes concentrated in degrees $0, \ldots, n$. Here N_n^* is defined in the obvious way by mimicking the definition of the usual conormalisation N^*. In particular for a cosimplicial abelian group $A^\bullet$ we have that $\mathrm{N}_n^*(A_{[n]}) \cong \tau^n(\mathrm{N}^*(A^\bullet))$.

Notice that, as with the usual Dold-Kan correspondence, for a truncated cosimplicial abelian group A in $\mathrm{Ab}^{\Delta[n]}$ we have an isomorphism $\lim^0_{\Delta[n]} A \cong \mathrm{H}^0(\mathrm{N}_n^*(A))$. By a universal δ-functor argument, H^p is the pth left derived functor of H^0 in $\mathrm{Ch}^{[n]}$. The equivalence of abelian categories implies therefore that $\lim^p_{\Delta[n]} A \cong \mathrm{H}^p(\mathrm{N}_n^*(A))$. Collecting these results proves our lemma. □

Proposition 4.3. *If $X^\bullet$ is well above the diagonal at the E^1 page then*

(i) *The homotopy and homology Bousfield-Kan spectral sequences associated to $X^\bullet$ converge strongly to $\pi_*(\mathrm{hoTot}\, X^\bullet)$ and $\mathrm{H}_*(\mathrm{hoTot}\, X^\bullet)$;*
(ii) *The homotopy and homology Bousfield-Kan spectral sequences associated to $\Pi^\bullet(X_{[n]})$ converge strongly to $\pi_*(\mathrm{hoTot}^n\, X^\bullet)$ and $\mathrm{H}_*(\mathrm{hoTot}^n\, X^\bullet)$;*
(iii) $\mathrm{H}_*(\mathrm{hoTot}\, X^\bullet) \cong \varprojlim \mathrm{H}_*(\mathrm{hoTot}^n\, X^\bullet)$.

Proof. (i) Since $X^\bullet$ is well above the diagonal at E^1 it is also well above the diagonal at E^2. Then the statement we want is exactly the content of the results in [6, Proposition IX.5.7] and [5, Theorem 3.2].

(ii) By Lemma 4.2 we have that $\Pi^\bullet(X_{[n]})$ is well above the diagonal at E^2. Moreover $\mathrm{hoTot}(\Pi^\bullet(X_{[n]})) \simeq \mathrm{hoTot}^n\, X^\bullet$. Thus (ii) follows from the same argument as in (i).

(iii) By the hypothesis and Lemma 4.2 we have that the connectivity of the map of spectral sequences between the E^2 pages of the homotopy spectral sequence for $X^\bullet$ and $\Pi^\bullet(X_{[n]})$ tends to infinity with n. By the convergence of these homotopy spectral sequences this implies that the connectivity of the map $\mathrm{hoTot}\, X^\bullet \to \mathrm{hoTot}^n\, X^\bullet$ tends to infinity with n. Therefore the same is true for the homologies. □

The following is part of the content of [29, Corollary 7.4].

Proposition 4.4 (Sinha)**.** *For $d \geq 4$, $\mathcal{K}_d^\bullet$ is well above the diagonal at the E^1 page.*

The last result of this section is a comparison theorem for Bousfield-Kan spectral sequences of cosimplicial replacements of diagrams which are connected by a left cofinal functor (see end of Section 2.1). We could not find a proof for this folklore result in the literature so we include one here. The proof below is due to W. Dwyer (P. Bousfield has also given us another proof).

Proposition 4.5. *Let $\theta\colon I \to J$ be a functor between* finite *categories and let $F\colon J \to \mathrm{Top}$ be a J-diagram of (pointed) spaces. If θ is left cofinal then both homotopy and rational homology Bousfield-Kan spectral sequences associated to the cosimplicial replacements $\Pi^\bullet F$ and $\Pi^\bullet(\theta^* F)$ agree from the E^2 pages.*

Proof. The proof mimicks that of [6, Proposition XI.9.2] and we follow most of the notation from there. Recall from the proof of Proposition 3.3 the simplicial CDGA associated to a contravariant functor of CDGAs. The Kunneth quasi-isomorphism

$$\otimes_\bullet A_{PL}(F) \xrightarrow{\simeq} A_{PL}(\Pi^\bullet F)$$

implies that the rational homology Bousfield-Kan spectral sequence for $\Pi^\bullet F$ coincides from the E_2 term with the spectral sequence of the double complex $\mathrm{N}_*(\otimes_\bullet A_{PL}(F))$, and we have an analogous result for $\Pi^\bullet \theta^*(F)$.

Define a bisimplicial commutative graded algebra (CGA)

$$\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta)$$

with the (n, q) term defined by

$$\otimes_{n,q}(\mathrm{H}^*(F), \theta) = \otimes_{(u,v,\gamma)} \mathrm{H}^*(F(j_0))$$

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where the tensor product is taken over

$$u = (i_0 \overset{\alpha_1}{\leftarrow} \cdots \overset{\alpha_n}{\leftarrow} i_n) \in \mathrm{N}_n(I), \quad v = (j_0 \overset{\beta_1}{\leftarrow} \cdots \overset{\beta_q}{\leftarrow} j_q) \in \mathrm{N}_q(J), \quad \gamma\colon \theta(i_0) \to j_q.$$

The faces and degeneracies in both directions are obvious generalizations of those in the proof of Proposition 3.3.

We can take the normalization of this bisimplicial CGA with respect to the first ($i = 1$) or the second ($i = 2$) simplicial degree to get a simplicial chain complex

$$\mathrm{N}^{(i)}_* \left(\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta)\right).$$

In the first direction we have

$$\left(\mathrm{N}^{(1)}_*(\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta))\right)_q = \otimes_{v \in \mathrm{N}_q(J)} \mathrm{N}_* \left(\otimes_{(u,\gamma)\in \mathrm{N}_\bullet(F\downarrow j_q)} \mathrm{H}^*(F(j_0))\right).$$

Denote by $\mathrm{H}^*(F(j_0))_\bullet$ the constant cosimplicial CGA which consists of the ring $\mathrm{H}^*(F(j_0))$ in each degree with the identity maps as faces and degeneracies. The category of simplicial CGAs is a simplicial model category [14, II.3 and II.5.2.(3)]. So we have the simplicial CGA

$$\mathrm{N}_\bullet(F \downarrow j_q) \otimes \mathrm{H}^*(F(j_0))_\bullet \cong \otimes_{(u,\gamma)\in \mathrm{N}_\bullet(F\downarrow j_q)} \mathrm{H}^*(F(j_0),$$

and by cofinality, $\mathrm{N}_\bullet(F \downarrow j_q)$ is weakly equivalent to the simplicial set $*_\bullet$ consisting of the singleton in each degree. Using [14, Proposition II.3.4.] we get a weak equivalence of simplicial CGAs

$$\mathrm{N}_\bullet(F \downarrow j_q) \otimes \mathrm{H}^*(F(j_0))_\bullet \simeq *_\bullet \otimes \mathrm{H}^*(F(j_0))_\bullet = \mathrm{H}^*(F(j_0))_\bullet$$

and deduce a weak equivalence of simplicial chain complexes

$$\mathrm{N}^{(1)}_*(\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta)) \simeq \otimes_\bullet \mathrm{H}^*(F) \tag{4.1}$$

where each $\mathrm{H}^*(F(j_0))$ is a chain complex concentrated in degree 0.

In the other direction,

$$\left(\mathrm{N}^{(2)}_*(\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta))\right)_n = \otimes_{u\in \mathrm{N}_n(I)} \mathrm{N}_* \left(\otimes_\bullet \mathrm{H}^*(\theta(i_0) \downarrow F)\right),$$

where $\theta(i_0) \downarrow F$ is the composite

$$\theta(i_0) \downarrow J \longrightarrow J \xrightarrow{F} \mathrm{Top}.$$

Here $\theta(i_0) \downarrow J$ is the undercategory (denoted by $J\backslash\theta(i_0)$ in [6]). This undercategory has the identity map at $\theta(i_0)$ as an initial object. It is easy to deduce, by an extra degeneracy or a spectral sequence argument, that $\otimes_\bullet \mathrm{H}^*(\theta(i_0) \downarrow F)$ is weakly equivalent to the constant simplicial CGA $\mathrm{H}(F(\theta(i_0))_\bullet$. We then have a weak equivalence of simplicial chain complexes

$$\mathrm{N}^{(2)}_*(\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta)) \simeq \otimes_\bullet \mathrm{H}^*(\theta^* F) \tag{4.2}$$

where each $\mathrm{H}^*(\theta^* F(i_0))$ is a chain complex concentrated in degree 0.

The left hand sides of the weak equivalences (4.1) and (4.2) are the E_1 page of spectral sequences computing the homology of the totalization of the double complex obtained as the double normalization of $\otimes_{\bullet,\bullet}(\mathrm{H}^*(F), \theta)$. Moreover these weak equivalences show that the homology of the E_1 pages is concentrated on a single line. Hence both of these spectral sequences collapse at E_2, and since they converge to the same thing, these E_2 pages are isomorphic. We deduce that the homologies of the normalizations of $\otimes_\bullet \mathrm{H}^*(F)$ and $\otimes_\bullet \mathrm{H}^*(\theta^* F)$ are isomorphic. Those are exactly the E_2 pages of the homology Bousfield Kan spectral sequences of $\Pi^\bullet F$ and $\Pi^\bullet \theta^* F$.

The proof for the homotopy spectral sequence is similar. □

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5. Categories of fans, fanic diagrams, and truncated cosimplicial objects

In this section we introduce a sequence of finite categories $\Phi[n]$ that we call categories of *n-fans*. They will serve in the next section as shapes of certain *fanic diagrams* associated to morphisms of operads, generalizing the Gerstenhaber-Voronov cosimplicial object associated to a multiplicative operad. We also construct a left cofinal functor $\phi_n \colon \Phi[n] \to \Delta[n]$.

Recall that a *planar tree* is an isotopy class of an embedding of the realization of a contractible finite 1-dimensional simplicial complex in the plane. In particular a planar tree consists of a finite set of *vertices* and *edges*. The *valence* of a vertex is the number of edges ending in that vertex. A *leaf* is a vertex of valence 1. The embedding in the plane induces a clockwise cyclic order on the leaves.

Definition 5.1.
- For a natural number n, an n*-fan* is a planar tree with a distinguished vertex called the *bead* such that each vertex, except maybe the bead, is of valence different from 2 and with $n+1$ leaves other than the bead which are labeled in the clockwise cyclic order by $0, 1, \ldots, n$. The leaf labeled 0 is called the *root*. The bead and vertices which are not leaves are called the *non-labeled vertices*.
- Define a partial order on the set of n-fans by declaring that $T \leq T'$ if the n-fan T' is obtained from the n-fan T by contracting some edges connecting non-labeled vertices and where the bead in T' is the vertex obtained by contracting the subtree of T containing its bead.
- The category corresponding to such a poset of n-fans is called the n*-fanic category* and is denoted by $\Phi[n]$. Thus there is a unique morphism $T \to T'$ if and only if $T \leq T'$.
- A diagram shaped on the fanic category $\Phi[n]$ is called an n*-fanic diagram*.

Example 5.2. Figure 1 gives examples of an 8-fan in (a) and of two 3-fans in (b) and (c). The bead is the vertex pictured as a small circle. The left-hand side of Figure 2 represents the category $\Phi[2]$.

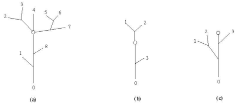

Figure 1. Three examples of fans

Remark 5.3. The name "fan" is motivated by the fact that one can define a functor from fans to rooted trees which "opens" them along the edge emanating from the bead, much as one opens a fan. For more details on this, see [21]. The realization of $\Phi[n]$ is homeomorphic to the barycentric subdivision of the n-dimensional cyclohedron C_n introduced by Bott and Taubes in [4].

Definition 5.4. We say that a n-fan T is i*-separated* for $i \in [n]$ if either
- the bead is not a leaf and the shortest path in T joining leaves i and $(i+1) \mod (n+1)$ goes through the bead, or
- the bead is a leaf and it is between leaves i and $(i+1) \mod (n+1)$ in the clockwise cyclic order.

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Example 5.5. In Example 5.2 the fans are i-separated for

(a) $i = 1, 3, 4, 7$;
(b) $i = 0, 2$;
(c) $i = 2$.

Let $\mathcal{P}_0([n])$ be the category whose objects are non-empty subsets of $[n] = \{0, \ldots, n\}$ and whose morphisms are inclusions. Define a functor

$$\theta_n \colon \Phi[n] \longrightarrow \mathcal{P}_0([n])$$

by setting, for an n-fan T,

$$\theta_n(T) := \{i \in [n] : T \text{ is } i\text{-separated}\}.$$

It is immediate that θ_n is an order-preserving map between the two posets and hence a functor.

Lemma 5.6. *The functor $\theta_n \colon \Phi[n] \longrightarrow \mathcal{P}_0([n])$ is left cofinal.*

Proof. It is easy to see that for any object $S \in \mathcal{P}_0([n])$ the overcategory $\theta_n \downarrow S$ has a terminal object and is therefore contractible. □

Recall the functor $\mathcal{G}_n \colon \mathcal{P}_0([n]) \to \Delta[n]$ from [29, Definition 6.3], defined as follows. For a non-empty subset $S \subset [n]$ consider the only order preserving bijection $f_S \colon S \xrightarrow{\cong} [(\#S) - 1]$. Define $\mathcal{G}_n(S) = [(\#S) - 1]$ and for a morphism $j \colon S \hookrightarrow T$ in $\mathcal{P}_0([n])$ let $\mathcal{G}_n(j)$ be the composite $f_T \circ j \circ f_S^{-1}$. It turns out that $\mathcal{G}_n$ is left cofinal [29, Theorem 6.7].

Now let ϕ_n be the composite

$$\phi_n := \mathcal{G}_n \theta_n \colon \Phi[n] \longrightarrow \Delta[n]. \tag{5.1}$$

Theorem 5.7. *The functor $\phi_n : \Phi[n] \to \Delta[n]$ is left cofinal.*

Proof. This is immediate from Lemma 5.6 and the cofinality of $\mathcal{G}_n$ from [29, Theorem 6.7]. □

Example 5.8. Figure 2 gives the shapes of $\Phi[2]$, $\mathcal{P}_0([2])$, and $\Delta[2]$ and the functors θ_2 and $\mathcal{G}_2$.

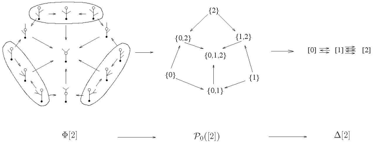

FIGURE 2. The functor ϕ_2

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6. The fanic diagram associated to a morphism of nonsymmetric operads

Recall from Section 2.2 the construction that associates a cosimplicial object to a multiplicative operad. Here we generalize this construction by associating a fanic diagram to any morphism of nonsymmetric operads.

Let $\mu\colon \mathcal{R} \to \mathcal{M}$ be a morphism of nonsymmetric operads in a symmetric monoidal category $\mathcal{C}$. Recall the category $\Phi[n]$ of n-fans defined in Section 5. We now describe a fanic diagram

$$\widehat{\mathcal{M}}_{\{n\}}\colon \Phi[n] \longrightarrow \mathcal{C}$$

which will depend on μ even if only $\mathcal{M}$ appears in the notation. We first define the value of the diagram $\widehat{\mathcal{M}}_{\{n\}}$ on objects. Let $T \in \Phi[n]$ be an n-fan. Orient each edge of T in the unique way such that its origin is on the shortest path joining the root to the end of that edge. For a vertex v other than the root let $|v|$ be the number of edges emanating from v. Since v has exactly one incoming edge, we have $|v| = \text{valence}(v) - 1$.

Recall that the leaves other than the bead of the fan T are labeled and that the other vertices, including the bead, are called the non-labeled vertices. For any non-labeled vertex v of T set

$$\mathcal{M}(T:v) := \begin{cases} \mathcal{M}(|v|), & \text{if } v \text{ is the bead;} \\ \mathcal{R}(|v|), & \text{if } v \text{ is not the bead.} \end{cases}$$

Set

$$\widehat{\mathcal{M}}_{\{n\}}(T) := \bigotimes_v \mathcal{M}(T:v)$$

where the monoidal product is taken over all the non-labeled vertices v of T.

To define $\widehat{\mathcal{M}}_{\{n\}}$ on morphisms, let e be an edge of T between two non-labeled vertices x and y such that e is oriented from x to y. Since x is not the root there is a single edge that ends in x. Label this edge 0 and label all the edges emanating from x with $1, \ldots, |x|$ such that they appear in the clockwise order. Hence the edge e is assigned some label $1 \le i \le |x|$. Let $\overline{T} := T/e$ be the fan obtained by contracting the edge e and let $\overline{e}$ be the vertex in $\overline{T}$ corresponding to that contracted edge. By definition, $\overline{e}$ is the bead of $\overline{T}$ if and only if x or y is the bead of T. Notice also that the non-labeled vertices v other than x and y in T are in bijection with the non-labeled vertices of $\overline{T}$ other than $\overline{e}$. The operadic structures and the morphism μ induce an obvious insertion map

$$\circ_i\colon \mathcal{M}(T:x) \otimes \mathcal{M}(T:y) \longrightarrow \mathcal{M}(\overline{T}:\overline{e}).$$

We define the morphism $\widehat{\mathcal{M}}_{\{n\}}(T \to T/e)$ to be the composite

$$\widehat{\mathcal{M}}_{\{n\}}(T) \cong \mathcal{M}(T:x) \otimes \mathcal{M}(T:y) \otimes \bigotimes_{v \neq x,y} \mathcal{M}(T:v) \stackrel{\circ_i \otimes \mathrm{id}}{\longrightarrow} \mathcal{M}(\overline{T}:\overline{e}) \otimes \bigotimes_{v \neq \overline{e}} \mathcal{M}(\overline{T}:v) \cong \widehat{\mathcal{M}}_{\{n\}}(\overline{T}).$$

Proposition 6.1. *Let $n \ge 0$ and let $\mathcal{C}$ be a symmetric monoidal category.*

(1) *For any morphism $\mu\colon \mathcal{R} \to \mathcal{M}$ of nonsymmetric operads in $\mathcal{C}$ the above construction gives a well-defined functor*

$$\widehat{\mathcal{M}}_{\{n\}}\colon \Phi[n] \longrightarrow \mathcal{C}.$$

(2) *This construction is functorial, i.e. any commutative square of nonsymmetric operads*

$$\begin{array}{ccc} \mathcal{R} & \stackrel{\mu}{\longrightarrow} & \mathcal{M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{R}' & \stackrel{\mu'}{\longrightarrow} & \mathcal{M}' \end{array} \tag{6.1}$$

induces a natural transformation $\widehat{g}\colon \widehat{\mathcal{M}}_{\{n\}} \longrightarrow \widehat{\mathcal{M}'}_{\{n\}}$.

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Proof. For (1), we have already defined $\widehat{\mathcal{M}}_{\{n\}}$ on objects and contractions of one edge. Since any morphism in $\Phi[n]$ is a composition of such contractions, we only have to check that the image under $\widehat{\mathcal{M}}_{\{n\}}$ of a composition does not depend on the order in which we contract the edges. For this, it is enough to check that if e_1 and e_2 are two distinct edges between non-labeled vertices in an n-fan T, then

$$\widehat{\mathcal{M}}_{\{n\}}(T/e_1 \to T/\{e_1,e_2\}) \circ \widehat{\mathcal{M}}_{\{n\}}(T \to T/e_1) = \widehat{\mathcal{M}}_{\{n\}}(T/e_2 \to T/\{e_1,e_2\}) \circ \widehat{\mathcal{M}}_{\{n\}}(T \to T/e_2).$$

This is an elementary check and is left to the reader.

For (2), given an object T of $\Phi[n]$, set $\widehat{g}(T) := \bigotimes_v \widehat{g}(T:v)$ where the tensor product is taken over the non-labeled vertices v of T and

$$\widehat{g}(T:v) := \begin{cases} g(|v|), & \text{if } v \text{ is the bead;} \\ f(|v|), & \text{if } v \text{ is not the bead.} \end{cases}$$

This defines the desired natural transformation. □

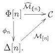
Suppose moreover that the category $\mathcal{C}$ is equipped with a certain class of morphisms called *weak equivalences*. This induces a class of weak equivalences on diagrams in $\mathcal{C}$ by declaring that a natural transformation between two diagrams is a weak equivalence if the map associated to each object of the indexing category is a weak equivalence. In particular this induces a class of weak equivalences of operads in $\mathcal{C}$.

Proposition 6.2. *Suppose that the symmetric monoidal category is equipped with a class of morphisms called weak equivalences that is stable under $\otimes$ and contains all isomorphisms. If the morphisms of nonsymmetric operads f and g in the commutative square (6.1) are weak equivalences then the natural transformation $\widehat{g}$ is also a weak equivalence.*

Proof. Follows from the definition of $\widehat{g}$ and the fact that the class of weak equivalences is stable under tensor product and contains the reordering isomorphisms. □

Lastly, we explain in which sense the above construction is a generalization of the cosimplicial object associated to a multiplicative nonsymmetric operad. Recall the functor $\phi_n : \Phi[n] \to \Delta[n]$ from Equation (5.1) which is left cofinal by Theorem 5.7. The proof of the following is straightforward.

Theorem 6.3. *Let $\mu: \mathcal{ASS} \to \mathcal{M}$ be a multiplicative nonsymmetric operad. Let $\mathcal{M}_{[n]} : \Delta[n] \to \mathcal{C}$ be the nth truncation of the associated cosimplicial object $\mathcal{M}^\bullet$ and let $\widehat{\mathcal{M}}_{\{n\}}$ be the n-fanic diagram asscoiated to the morphism μ. Then the following diagram commutes:*



Corollary 6.4. *Let $\mathcal{M}$ be a multiplicative operad of spaces, let $\mathcal{M}^\bullet$ be the associated cosimplicial space and let $\widehat{\mathcal{M}}_{\{n\}}$ be the associated n-fanic diagram for $n \geq 0$. Then there is a homotopy equivalence*

$$\operatorname*{holim}_{\Phi[n]} \widehat{\mathcal{M}}_{\{n\}} \simeq \mathrm{hoTot}^n(\mathcal{M}^\bullet).$$

Proof. This is a consequence of Theorem 6.3 and Theorem 5.7. □

Remark 6.5. Notice that all the results of this section can be easily dualized for cooperads. In particular, to a morphism of cooperads $\mu^* : \mathcal{M}^* \to \mathcal{R}^*$ in $\mathcal{C}$ we can associate a *cofanic* diagram $\widehat{\mathcal{M}^*}_{\{n\}} : \Phi[n]^{op} \to \mathcal{C}$ where $(-)^{op}$ means the opposite category. We leave it as an exercise for the reader to state and prove these dual results.

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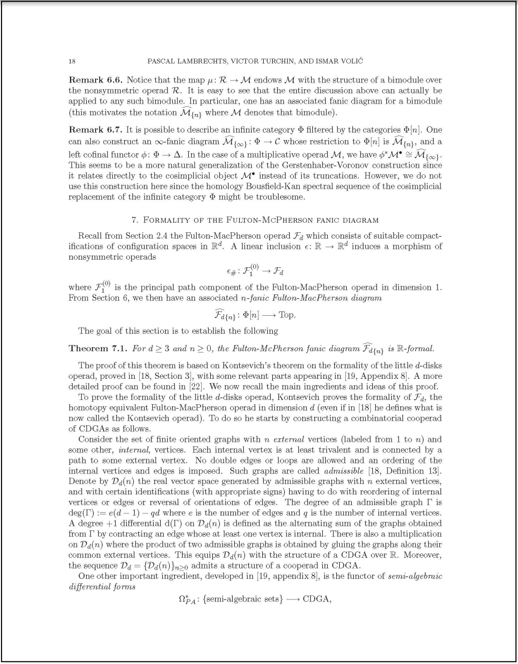

Remark 6.6. Notice that the map $\mu\colon \mathcal{R} \to \mathcal{M}$ endows $\mathcal{M}$ with the structure of a bimodule over the nonsymmetric operad $\mathcal{R}$. It is easy to see that the entire discussion above can actually be applied to any such bimodule. In particular, one has an associated fanic diagram for a bimodule (this motivates the notation $\widehat{\mathcal{M}}_{\{n\}}$ where $\mathcal{M}$ denotes that bimodule).

Remark 6.7. It is possible to describe an infinite category Φ filtered by the categories $\Phi[n]$. One can also construct an ∞-fanic diagram $\widehat{\mathcal{M}}_{\{\infty\}}\colon \Phi \to \mathcal{C}$ whose restriction to $\Phi[n]$ is $\widehat{\mathcal{M}}_{\{n\}}$, and a left cofinal functor $\phi\colon \Phi \to \Delta$. In the case of a multiplicative operad $\mathcal{M}$, we have $\phi^*\mathcal{M}^\bullet \cong \widehat{\mathcal{M}}_{\{\infty\}}$. This seems to be a more natural generalization of the Gerstenhaber-Voronov construction since it relates directly to the cosimplicial object $\mathcal{M}^\bullet$ instead of its truncations. However, we do not use this construction here since the homology Bousfield-Kan spectral sequence of the cosimplicial replacement of the infinite category Φ might be troublesome.

7. Formality of the Fulton-McPherson fanic diagram

Recall from Section 2.4 the Fulton-MacPherson operad $\mathcal{F}_d$ which consists of suitable compactifications of configuration spaces in $\mathbb{R}^d$. A linear inclusion $\epsilon\colon \mathbb{R} \to \mathbb{R}^d$ induces a morphism of nonsymmetric operads

$$\epsilon_\#\colon \mathcal{F}_1^{(0)} \to \mathcal{F}_d$$

where $\mathcal{F}_1^{(0)}$ is the principal path component of the Fulton-MacPherson operad in dimension 1. From Section 6, we then have an associated *n-fanic Fulton-MacPherson diagram*

$$\widehat{\mathcal{F}}_{d\{n\}}\colon \Phi[n] \longrightarrow \mathrm{Top}.$$

The goal of this section is to establish the following

Theorem 7.1. *For $d \geq 3$ and $n \geq 0$, the Fulton-McPherson fanic diagram $\widehat{\mathcal{F}}_{d\{n\}}$ is $\mathbb{R}$-formal.*

The proof of this theorem is based on Kontsevich's theorem on the formality of the little d-disks operad, proved in [18, Section 3], with some relevant parts appearing in [19, Appendix 8]. A more detailed proof can be found in [22]. We now recall the main ingredients and ideas of this proof.

To prove the formality of the little d-disks operad, Kontsevich proves the formality of $\mathcal{F}_d$, the homotopy equivalent Fulton-MacPherson operad in dimension d (even if in [18] he defines what is now called the Kontsevich operad). To do so he starts by constructing a combinatorial cooperad of CDGAs as follows.

Consider the set of finite oriented graphs with n *external* vertices (labeled from 1 to n) and some other, *internal*, vertices. Each internal vertex is at least trivalent and is connected by a path to some external vertex. No double edges or loops are allowed and an ordering of the internal vertices and edges is imposed. Such graphs are called *admissible* [18, Definition 13]. Denote by $\mathcal{D}_d(n)$ the real vector space generated by admissible graphs with n external vertices, and with certain identifications (with appropriate signs) having to do with reordering of internal vertices or edges or reversal of orientations of edges. The degree of an admissible graph Γ is $\deg(\Gamma) := e(d-1) - qd$ where e is the number of edges and q is the number of internal vertices. A degree $+1$ differential $\mathrm{d}(\Gamma)$ on $\mathcal{D}_d(n)$ is defined as the alternating sum of the graphs obtained from Γ by contracting an edge whose at least one vertex is internal. There is also a multiplication on $\mathcal{D}_d(n)$ where the product of two admissible graphs is obtained by gluing the graphs along their common external vertices. This equips $\mathcal{D}_d(n)$ with the structure of a CDGA over $\mathbb{R}$. Moreover, the sequence $\mathcal{D}_d = \{\mathcal{D}_d(n)\}_{n\geq 0}$ admits a structure of a cooperad in CDGA.

One other important ingredient, developed in [19, appendix 8], is the functor of *semi-algebraic differential forms*

$$\Omega^*_{PA}\colon \{\text{semi-algebraic sets}\} \longrightarrow \mathrm{CDGA},$$

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mimicking the de Rham functor of smooth differential forms on smooth manifolds, where a semi-algebraic set is a subset of $\mathbb{R}^n$ defined by finite sets of polynomial inequalities and boolean operations. This functor Ω^*_{PA} is naturally quasi-isomorphic to the functor $A_{PL}(-;\mathbb{R})$. One fact which will be important to us is that if X is a semi-algebraic set of dimension $\dim(X) \leq m$ then $\Omega^i_{PA}(X) = 0$ for $i > m$.

Spaces $\mathcal{F}_d(n)$ are semi-algebraic manifolds and Konsevich's idea is to assign to each admissible graph $\Gamma \in \mathcal{D}_d(n)$ of degree r a certain semi-algebraic differential form $I(\Gamma) = \omega_\Gamma \in \Omega^r_{PA}(\mathcal{F}_d(n))$. This defines a CDGA morphism

$$I_n \colon \mathcal{D}_d(n) \longrightarrow \Omega^*_{PA}(\mathcal{F}_d(n)).$$

On the other hand, it is easy to construct an explicit CDGA morphism

$$\bar{I}_n \colon \mathcal{D}_d(n) \longrightarrow \mathrm{H}^*(\mathcal{F}_d(n);\mathbb{R})$$

that sends to 0 any admissible graphs with at least one internal vertex and that sends the admissible graph with a unique edge joining the external vertices i and j, for $1 \leq i < j \leq n$, to the generator g_{ij} in the usual presentation of the cohomology of the configuration space

$$\mathrm{H}^*(\mathcal{F}_d(n);\mathbb{R}) \cong \bigwedge(\{g_{ij} : 1 \leq i < j \leq n\})/\sim.$$

The exact presentation for the three equivalence relations $\sim$ (one of which is the Arnold, or three-term, relation) can be found, for example, in [29, Section 7].

It can be proved that $\bar{I}_n$ is a quasi-isomorphism and, since I_n is surjective on the indecomposables in cohomology, it is also a quasi-isomorphism. Finally, it would be nice if these were quasi-isomorphisms of cooperads, but they are not because the contravariant functor Ω^*_{PA} is not monoidal and so $\Omega^*_{PA}(\mathcal{F}_d)$ does not inherit the structure of a cooperad. Nevertheless $I = \{I_n\}_{n\geq 0}$ is almost a quasi-isomorphism of cooperads. More precisely, we have the following

Theorem 7.2 ([22]). *There is a commutative diagram*

$$\begin{array}{ccc}
\mathcal{D}_d(n) & \xrightarrow{\ \phi\ } & \mathcal{D}_d(k) \otimes \mathcal{D}_d(n_1) \otimes \cdots \otimes \mathcal{D}_d(n_k) \\
 & & \downarrow{\scriptstyle \simeq\ I_k \otimes I_{n_1} \otimes \cdots \otimes I_{n_k}} \\
{\scriptstyle I_n\ \simeq}\downarrow & & \Omega^*_{PA}(\mathcal{F}_d(k)) \otimes \Omega^*_{PA}(\mathcal{F}_d(n_1)) \otimes \cdots \otimes \Omega^*_{PA}(\mathcal{F}_d(n_k)) \\
 & & \downarrow{\scriptstyle \simeq\ Kunneth} \\
\Omega^*_{PA}(\mathcal{F}_d(n)) & \xrightarrow{\ \Omega^*_{PA}(\mu)\ } & \Omega^*_{PA}(\mathcal{F}_d(k) \times \mathcal{F}_d(n_1) \times \cdots \times \mathcal{F}_d(n_k))
\end{array} \tag{7.1}$$

where μ and ϕ are the (co)operadic structure maps on $\mathcal{F}_d$ and $\mathcal{D}_d$ and the vertical maps are quasi-isomorphisms.

On the other hand, $\{\bar{I}_n \colon \mathcal{D}_d(n) \xrightarrow{\simeq} \mathrm{H}^*(\mathcal{F}_d(n);\mathbb{R})\}_{n\geq 0}$ is a quasi-isomorphism of cooperads. This, combined with Theorem 7.2, is what we mean when we say that the Fulton-MacPherson operad is formal over $\mathbb{R}$. Notice that if we work dually at the level of chains, we can use a monoidal functor to get genuine formality in the category of chain complexes.

In order to prove the formality of the Fulton-MacPherson fanic diagram, we will prove the formality of the morphism $\epsilon_\# \colon \mathcal{F}_1^{(0)} \to \mathcal{F}_d$. The key result we need is Lemma 7.3 below. To explain this, let $\mathcal{ASS} = \{\mathbb{R}\}_{n\geq 0}$ be the coassociative cooperad in CDGA. In degree 0, the vector space $\mathcal{D}_d(n)$ is generated by the single admissible graph with n external vertices and no edges. These isomorphisms $\mathcal{D}_d(n)^0 \cong \mathbb{R}$ induces a morphism of nonsymmetric cooperads in CDGA

$$\epsilon \colon \mathcal{D}_d \longrightarrow \mathcal{ASS}.$$

Let $\eta_n \colon \mathbb{R} \to \Omega^*_{PA}(\mathcal{F}_1^{(0)}(n))$ be the inclusion of constants in degree 0.

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Lemma 7.3. *For $d \geq 3$, diagram*

$$
\begin{array}{ccccc}
H^*(\mathcal{F}_d(n);\mathbb{R}) & \xleftarrow[\simeq]{I} & \mathcal{D}_d(n) & \xrightarrow[\simeq]{I_n} & \Omega^*_{PA}(\mathcal{F}_d(n)) \\
\downarrow{\scriptstyle H^*(\epsilon_\#)} & & \downarrow{\scriptstyle \epsilon_n} & & \downarrow{\scriptstyle \Omega^*_{PA}(\epsilon_\#)} \\
H^*(\mathcal{F}_1^{(0)}(n);\mathbb{R}) & \xleftarrow{\cong} & \mathbb{R} & \xrightarrow[\eta_n]{\simeq} & \Omega^*_{PA}(\mathcal{F}_1^{(0)}(n))
\end{array}
$$

commutes.

Proof. The left square is clearly commutative. For the right square in case $n \leq 1$, $\mathcal{D}_d(n)$ is concentrated in degree 0 and diagram clearly commutes. So assuming $n \geq 2$, let $\Gamma \in \mathcal{D}_d(n)$ be an admissible graph of positive degree with n external vertices. Assume first that Γ has no isolated vertices, i.e. each external vertex is an endpoint of at least one edge. Let q be the number of internal vertices of Γ and recall that every internal vertex has valence at least 3. Then the number e of edges of Γ satisfies

$$e \geq \frac{1}{2}(n + 3q).$$

Therefore

$$\deg(I(\Gamma)) = \deg(\Gamma) = e(d-1) - qd \geq \frac{1}{2}(n+3q)(d-1) - qd = q\left(\frac{d-3}{2}\right) + n\left(\frac{d-1}{2}\right).$$

Since $d \geq 3$ this implies that $\deg(I(\Gamma)) \geq n$. On the other hand, we have $\dim(\mathcal{F}_1^{(0)}(n)) = n - 2$ and so

$$\deg(I(\Gamma)) > \dim(\mathcal{F}_1^{(0)}(n)).$$

Since for any semi-algebraic set X, $\Omega^i_{PA}(X) = 0$ if $i > \dim(X)$, we deduce that

$$\Omega^*_{PA}(\epsilon)(I(\Gamma)) = 0 \in \Omega^*_{PA}(\mathcal{F}_1^{(0)}(n)).$$

Now suppose Γ has only $j < n$ external vertices which are not isolated. Let $\sigma\colon \mathcal{D}_d(j) \to \mathcal{D}_d(n)$ be the composite of codegeneracies σ_i inserting the $n-j$ isolated vertices. Hence there exists an admissible graph $\Gamma' \in \mathcal{D}_d(j)$ with no isolated vertex and such that $\Gamma = \sigma(\Gamma')$. Consider the map $s\colon \mathcal{F}_d(n) \to \mathcal{F}_d(j)$ obtained as the composition of $n-j$ codegeneracies s^i that forget the configuration points corresponding to these isolated vertices. Since (co)degeneracies are induced by the operadic structure and I is a map of (almost) cooperads, it is easy to check that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathcal{D}_d(j) & \xrightarrow[\simeq]{I} & \Omega^*_{PA}(\mathcal{F}_d(j)) & \xrightarrow{\epsilon^*_\#} & \Omega^*_{PA}(\mathcal{F}_1^{(0)}(j)) \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s^*} \\
\mathcal{D}_d(n) & \xrightarrow[\simeq]{I} & \Omega^*_{PA}(\mathcal{F}_d(n)) & \xrightarrow{\epsilon^*_\#} & \Omega^*_{PA}(\mathcal{F}_1^{(0)}(n))
\end{array}
$$

Since Γ' has no isolated vertices it follows from the previous case that $\epsilon^*_\#(I(\Gamma')) = 0$ and so

$$\epsilon^*_\#(I(\Gamma)) = \epsilon^*_\#(I(\sigma(\Gamma'))) = s^*(\epsilon^*_\#(I(\Gamma'))) = 0.$$

□

We are now ready for the proof of the formality of the Fulton-MacPherson fanic diagram.

Proof of Theorem 7.1. The fanic diagram $\widehat{\mathcal{F}}_{d\{n\}}$ induces a cofanic diagram of CDGAs

$$\Omega^*_{PA}(\widehat{\mathcal{F}}_{d\{n\}})\colon \Phi[n]^{op} \longrightarrow \text{CDGA}.$$

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We also have a diagram of CDGA cooperads

$$\begin{array}{ccc} \mathcal{D}_d & \xrightarrow{\epsilon} & \mathcal{ASS} \\ \simeq \downarrow I & & = \downarrow \\ H^*(\mathcal{F}_d;\mathbb{R}) & \xrightarrow{\epsilon^*} & H^*(\mathcal{F}_1^{(0)};\mathbb{R}) = \mathcal{ASS} \end{array}$$

which induces quasi-isomorphic cofanic diagrams

$$\widehat{\mathcal{D}_{d\{n\}}} \simeq \widehat{H^*(\mathcal{F}_d;\mathbb{R})}_{\{n\}} \colon \Phi[n]^{op} \longrightarrow \text{CDGA},$$

and of course the cofanic diagram $\widehat{H^*(\mathcal{F}_d;\mathbb{R})}_{\{n\}}$ is $\mathbb{R}$-formal.

We still need to construct a natural quasi-isomorphism

$$\widehat{I} \colon \widehat{\mathcal{D}_{d\{n\}}} \xrightarrow{\simeq} \Omega^*_{PA}(\widehat{\mathcal{F}_{d\{n\}}}).$$

For a fan $T \in \Phi[n]$ with bead b, and for a non-root vertex v recall that $|v| = \text{valence}(v) - 1$. Consider the morphism

$$I_{|b|} \otimes \otimes_{v \neq b} \eta_{|v|} \colon \widehat{\mathcal{D}_{d\{n\}}}(T) \cong \mathcal{D}_d(|b|) \otimes \otimes_{v\neq b}\mathbb{R} \longrightarrow \Omega^*_{PA}(\mathcal{F}_d(|b|)) \otimes \otimes_{v \neq b} \Omega^*_{PA}(\mathcal{F}_1^{(0)}(|v|))$$

and define $\widehat{I}(T)$ as the composite of this map with the Kunneth quasi-isomorphism

$$\Omega^*_{PA}(\mathcal{F}_d(|b|)) \otimes \otimes_{v\neq b} \Omega^*_{PA}(\mathcal{F}_1^{(0)}(|v|)) \xrightarrow{\simeq} \Omega_{PA}\left(\mathcal{F}_d(|b|) \times \prod_{v \neq b} \mathcal{F}_1^{(0)}(|v|)\right).$$

We need to show that $\widehat{I}$ is a natural transformation. Remember from Section 6 the definition of the image of a morphism by the fanic diagram associated to a morphism of (co)operads. Let e be an edge of T emanating from x and ending at y, let i be the label of the output e with respect to its origin x, and let $\overline{e}$ be the vertex in $\overline{T} = T/E$ corresponding to this contracted edge. Denote by v any vertex of T different from x and y or any vertex of $\overline{T}$ different from $\overline{e}$ (these two sets of vertices are the same). We have three cases: (i) $x = b \neq y$, (ii) $x \neq b = y$, and (iii) $x \neq b \neq y$. We will treat only case (i), the second case being completely analoguous and the third simpler. In case (i), $\overline{e}$ is the bead of $\overline{T}$. Consider the diagram

(7.2)
$$\begin{array}{ccccc} \mathcal{D}_d(|\overline{e}|) & & \xrightarrow[\simeq]{I_{|\overline{e}|}} & & \Omega^*_{PA}(\mathcal{F}_d(|\overline{e}|)) \\ \downarrow \circ_i & & & & \downarrow \Omega^*_{PA}(\circ_i) \\ \mathcal{D}_d(|x|) \otimes \mathcal{D}_d(|y|) & \xrightarrow[\simeq]{I_{|x|}\otimes I_{|y|}} & \Omega^*_{PA}(\mathcal{F}_d(|x|)) \otimes \Omega^*_{PA}(\mathcal{F}_d(|y|)) & \xrightarrow[\simeq]{\text{Kunneth}} & \Omega^*_{PA}(\mathcal{F}_d(|x|) \times \mathcal{F}_d(|y|)) \\ \downarrow \text{id}\otimes\epsilon & & \downarrow \text{id}\otimes\epsilon^* & & \downarrow (\text{id}\times\epsilon)^* \\ \mathcal{D}_d(|x|) \otimes \mathbb{R} & \xrightarrow[\simeq]{I_{|x|}\otimes \eta_{|y|}} & \Omega^*_{PA}(\mathcal{F}_d(|x|)) \otimes \Omega^*_{PA}(\mathcal{F}_1^{(0)}(|y|)) & \xrightarrow[\simeq]{\text{Kunneth}} & \Omega^*_{PA}(\mathcal{F}_d(|x|) \times \mathcal{F}_1^{(0)}(|y|)) \end{array}$$

The top rectangle is commutative as a consequence of the commutativity of diagram (7.1) and the definition of the insertion maps $\circ_i$ from the (co)operad structures. The left bottom square is commutative by Lemma 7.3, and the right bottom square is commutative by naturality of the Kunneth quasi-isomorphism.

Consider now the outermost square of (7.2) and tensor the left side by $\otimes_v \mathbb{R}$ and the right side by $\otimes_{|v|} \Omega^*_{PA}(\mathcal{F}_1^{(0)}(|v|))$. Applying once more the Kunneth quasi-isomorphism to the right side

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gives the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{D}}_{d\{n\}}(\overline{T}) & \xrightarrow[\simeq]{\widehat{I}(\overline{T})} & \Omega^*_{PA}(\widehat{\mathcal{F}}_{d\{n\}}(\overline{T})) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{D}}_{d\{n\}}(T) & \xrightarrow[\simeq]{\widehat{I}(T)} & \Omega^*_{PA}(\widehat{\mathcal{F}}_{d\{n\}}(T)) \end{array}$$

which is what we were after. □

Remark 7.4. Let us explain more precisely what we mean when we say that we cannot prove that the Kontsevich operad is formal as a multiplicative operad and why we need to go through fanic diagrams. Lemma 7.3 proves that the morphism $\epsilon_\#\colon \mathcal{F}_1^{(0)} \to \mathcal{F}_d$ is formal, so the weak equivalence (2.5) implies that we have a chain of quasi-isomorphisms of morphisms of (almost) cooperads between $A_{PL}(\mathcal{ASS}) \to A_{PL}(\mathcal{K}_d)$ and $\mathrm{H}^*(\mathcal{ASS};\mathbb{R}) \to \mathrm{H}^*(\mathcal{K}_d;\mathbb{R})$. The two cooperads $A_{PL}(\mathcal{K}_d)$ and $H^*(\mathcal{K}_d;\mathbb{R})$ are comultiplicative since $A_{PL}(\mathcal{ASS}) = \mathrm{H}^*(\mathcal{ASS};\mathbb{R}) = \{\mathbb{R}\}_{n\geq 0}$. The problem is that, in the chain of quasi-isomorphisms joining $A_{PL}(\mathcal{ASS})$ and $\mathrm{H}^*(\mathcal{ASS};\mathbb{R})$, nothing guarantees that all the intermediate CDGAs will be coassociative even if their homologies are. Therefore we cannot apply the dual of the Gerstenhaber-Voronov/McClure-Smith construction to the intermediate CDGAs. One way around this would be to use "simplicial up to homotopy" CDGAs but we wanted to avoid that. Instead, fanic diagrams allow us to work with strictly commutative diagrams.

8. Proof of the main theorem

In this section we finally prove Theorem 1.2. We begin with

Proposition 8.1. *For $d \geq 3$ and $n \geq 0$ the rational homology Bousfield-Kan spectral sequence of the cosimplicial replacement $\Pi^\bullet\mathcal{K}_{d[n]}$ of the nth truncation of Sinha's cosimplicial space $\mathcal{K}_d^\bullet$ collapses at the E^2 page.*

Proof. By Theorem 7.1, the Fulton-MacPherson fanic diagram $\widehat{\mathcal{F}}_{d\{n\}}$ is $\mathbb{R}$-formal. By diagram (2.5) and Proposition 6.2, we have an equivalence of diagrams $\widehat{\mathcal{F}}_{d\{n\}} \simeq \widehat{\mathcal{K}}_{d\{n\}}$ and by Theorem 6.3 we have $\widehat{\mathcal{K}}_{d\{n\}} = \phi_n^*(\mathcal{K}_{d[n]})$ where $\phi_n\colon \Phi[n] \to \Delta[n]$ is the left cofinal functor from Theorem 5.7.

Thus the finite diagram $\phi_n^*(\mathcal{K}_{d[n]})$ is $\mathbb{R}$-formal, and so is its cosimplicial replacement by Proposition 3.3. By Proposition 3.2, its homology spectral sequence collapses at the E^2 page. Since ϕ_n is left cofinal by Theorem 5.7, we deduce from Proposition 4.5 that the homology spectral sequence of $\Pi^\bullet\mathcal{K}_{d[n]}$ collapses at E^2 page. □

We are now ready for the proof of our main theorem. The trick will be to replace $\mathcal{K}_d^\bullet$ (which we do not know if it is formal) by an associated formal cosimplicial space $\Xi^\bullet$.

Proof of Theorem 1.2. We first recall some ideas from rational homotopy theory. Let CDGA_1 be the full subcategory of CDGA over $\mathbb{Q}$ of simply-connected CDGAs of finite type, and let DGL be the category of connected differential graded Lie algebras as in [10, §21 (f)]. As explained in [10, §22], there are two functors

$$\mathrm{CDGA}_1 \xrightarrow{\mathcal{L}} \mathrm{DGL} \xrightarrow{C^*} \mathrm{CDGA}$$

where $\mathcal{L}(A)$ is essentially the primitive part of the cobar on the dual coalgebra $\hom(A;\mathbb{Q})$ and $C^*(L)$ is the dual of the bar construction on the enveloping algebra of $L \in \mathrm{DGL}$. The only property of these functors of interest to us is that, for $A \in \mathrm{CDGA}_1$, $C^*(\mathcal{L}(A))$ is a Sullivan algebra quasi-isomorphic to A [10, §22 (e)]. In other words, $C^*(\mathcal{L}(-))$ can serve as a cofibrant replacement functor.

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We also have a *spatial realization* functor

$$|-|: \mathrm{CDGA} \to \mathrm{Top}$$

defined in [10, §17]. An important property of this functor is that a Sullivan algebra A is naturally weakly equivalent to $A_{PL}(|A|;\mathbb{Q})$ [10, §17 (d)].

Now consider the simplicial CDGA $H^*(\mathcal{K}_d^\bullet;\mathbb{Q})$ and define the cosimplicial space

$$\Xi^\bullet := |C^*(\mathcal{L}(H^*(\mathcal{K}_d^\bullet;\mathbb{Q})))|.$$

It is clear from the discussion above that this cosimplicial space is formal and has the same cohomology as $\mathcal{K}_d^\bullet$.

A more surprising property, arising from the fact that the cohomology algebras of configuration spaces in $\mathbb{R}^n$ are Koszul, is that the cosimplicial space $\Xi^\bullet$ is also coformal. This is not difficult to prove and details are in [1, Section 4], in particular Corollary 4.3, noticing that $\Xi^\bullet$ is exactly $|\chi^\bullet|$ in that paper. Also explained in that paper is the consequence that there exists an isomorphism of cosimplicial groups

$$\pi_*(\Xi^\bullet) \cong \pi_*(\mathcal{K}_d^\bullet) \otimes \mathbb{Q}. \tag{8.1}$$

We know by Proposition 4.4 that $\mathcal{K}_d^\bullet$ is well above the diagonal at the E^1 page and by (8.1) the same is true for $\Xi^\bullet$. Therefore the convergence results from Proposition 4.3 hold for both $\mathcal{K}_d$ and $\Xi^\bullet$.

Since $\Xi^\bullet$ is formal, the same is true for $\Pi^\bullet\Xi_{[n]}$, so the homology spectral sequence of that cosimplicial replacement collapses at the E^2 page. But this E^2 page is clearly isomorphic to the E^2 page of the homology spectral sequence of $\Pi^\bullet\mathcal{K}_{d[n]}$, since the two cosimplicial spaces have the same homology. Moreover, the second spectral sequence also collapses by Proposition 8.1, and both spectral sequences converge by Proposition 4.3. Therefore $\mathrm{H}_*(\mathrm{hoTot}^n\,\Xi^\bullet;\mathbb{Q}) \cong \mathrm{H}_*(\mathrm{hoTot}^n\,\mathcal{K}_d^\bullet;\mathbb{Q})$, and by Proposition 4.3(iii) this implies

$$\mathrm{H}_*(\mathrm{hoTot}\,\Xi^\bullet;\mathbb{Q}) \cong \mathrm{H}_*(\mathrm{hoTot}\,\mathcal{K}_d^\bullet;\mathbb{Q}). \tag{8.2}$$

By formality of $\Xi^\bullet$, its rational homology Bousfield-Kan spectral sequence also collapses at the E^2 page, which is isomorphic to the E^2 page of $\mathcal{K}_d^\bullet$. Since both these spectral sequences converge to the isomorphic terms of (8.2), we deduce that the homology spectral sequence of $\mathcal{K}_d^\bullet$ also collapses at the E^2 page. □

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Back to generalizations about mathematics

- The combination of rigor, abstractness, and vastness is what makes mathematics as beautiful as it is difficult;
- Because mathematics is so complex, it takes a long time for it to be created, written up, refereed, published, absorbed, reinterpreted, and used;
- Making bridges between the 5,000 subjects can be difficult because of our highly compartmentalized knowledge, and this is why mathematics is very collaborative;
- Because proofs are irrefutable, mathematics is permanent.