SOME PROBLEMS ARISING FROM HOMOTOPY-THEORETIC METHODS IN KNOT THEORY

ISMAR VOLIĆ, May 25, 2010

This is a partial list of some interesting questions that arose in the past decade or so from applications of homotopy-theoretic methods in knot and link theory. The ones I have in mind (because those are the ones I am familiar with) are manifold calculus of functors, cosimplicial spaces, and operad actions. (Some problems are also about configuration space integrals as they have proven to be a useful tool in this story.) There are doubtless others that I have left out. If you feel something should be added to this document, please let me know. It is quite likely that the chart on the last page is incomplete, that it requires more arrows, that some arrows shouldn't be there, that the list of open problems is incomplete, or even that some of the problems are incorrectly stated or have been solved. If this is the case, I'd like to hear about it so I can make corrections.

The first list of open problems arises naturally from existing literature. Thus the numbering of the problems correspond to the labels of the dotted arrows on the chart on the last page. The second list is just an assortment of other problems that came to mind. Neither list is in any particular order. Many of the problems are related (the list could probably be condensed) and answering one question from the list is likely to answer some of the others.

1. Open problems from the chart on the last page

(1) Both Budney [7] and Salvatore [38] have shown that the framed long knots in $\mathbb{R}^n$, $n > 3$, are double loop spaces, but from two very different points of view. Salvatore uses McClure-Smith cosimplicial machinery while Budney constructs an explicit action of the little discs operad (actually, Budney's construction works for classical knots as well). The question is if whether those two-fold loop space structures are compatible. (Also see (4) below.)

(2) Link maps, and in particular link maps of $\coprod_k \mathbb{R}$ in $\mathbb{R}^n$, i.e. homotopy string links of k components in $\mathbb{R}^n$ for any n, are studied in [31, 32] using manifold calculus. It would be extremely beneficial to recast the Habegger-Lin [17] classification result for homotopy string links in this language.

(3) Results in [53] can be interpreted as collapse of the Vassiliev cohomology spectral sequence (containing configuration spaces in $\mathbb{R}^3$) on the diagonal. The same spectral sequence is shown to collapse rationally in [26], but now the configurations are in $\mathbb{R}^n$, $n > 3$. Can one show that the original spectral sequence collapses everywhere rationally? Formality of configuration spaces still holds, so that should help.

(4) Sinha [43] showed that little discs operad acts on the space of knots modulo immersions in $\mathbb{R}^n$, $n \geq 3$. The question is whether this action is compatible with Budney's action on framed knots [7]. Since Salvatore [38] takes Sinha's work as input, answering (1) would also answer this in the affirmative.

(5) Koycheff [24] constructs cohomology classes on long knots in $\mathbb{R}^n$, $n \geq 3$, by using a Pontryagin-Thom construction on the same bundle considered in Bott-Taubes integration [5, 11, 54]. This

allows for the construction of classes over $\mathbb{Z}$ on the spaces of knots, which is a nice improvement. However, since Koytcheff's construction collapses the entire boundary of the total space of the bundle in question (spaces $\mathcal{K}[k,s;\mathcal{K},\mathbb{R}^n]$), it is hard to understand how his classes compare to the Bott-Taubes ones from [5, 11]. In particular, it would be nice to know whether Koytcheff's classes are of finite type in the case of classical knots. A variant of his construction which might be better suited for this comparison could come from identifying the boundaries of the various $\mathcal{K}[k,s;\mathcal{K},\mathbb{R}^n]$'s, rather than collapsing them.

(6) Budney and Cohen exhibit the rational homology of the space of classical long knots as a Poisson-Gerstenhaber algebra and give generators (they also give analogous results for mod p homology). Similarly, the homology of long knots (modulo immersions) in $\mathbb{R}^n$, $n>3$, is the Hochschild homology of the Poisson algebras operad [26]. It would be nice to explore the connections more. In particular, [26, 25] provides nice graph complex whose homology is the homology of knots; it might be nice to have a similar way of viewing the Budney-Cohen results.

(7) Both [51] and Kohno [21] describe integration techniques for constructing cohomology classes on braids in $\mathbb{R}^n$, $n\geq 3$. In [51], Bott-Taubes integrals are used while Kohno uses Chen integrals. It would be interesting to establish a connection between the two. The main difference is that in [51], braids are regarded as subspaces of embedding spaces, while in [21], they are thought of as loops of configuration spaces. One common thread they have is that they both describe some version of finite type invariants and their generalizations.

(8) In [11], Cattaneo, Cota-Ramusino, and Longoni construct a double complex which gives rise to cohomology classes on spaces of knots in $\mathbb{R}^n$, $n\geq 3$, via Bott-Taubes configuration space integrals. The conjecture is that all cohomology classes on knot spaces arise in this way. To show this, it would suffice to show that their complex is quasi-isomorphic to the E_1 page of the Vassiliev spectral sequence [49] (or the orthogonal spectral sequence from [3]).

(9) Related to the previous two questions, [51] gives two diagram complexes and cochain maps to the deRham complexes of braids and links. The conjecture is that these maps are quasi-isomorphisms (as should be the case with the corresponding map for long knots constructed in [11]). In case of braids, this should recover or at least tie into something that's already known by work of Cohen-Gitler [14, 13] and Kohno [21].

(10) In analogy with the cohomology spectral sequence converging to long knots in $\mathbb{R}^n$, $n>3$, which is shown to collapse rationally in [26], one has spectral sequences converging to string links and homotopy string links [32]. In principle, the collapse of those spectral sequences should not be too hard to show. The first page still consists of cohomology of configuration spaces, which are formal – this is the main ingredient in [26] – although for homotopy string links they are "partial" configuration spaces which may complicate the situation.

(11) It is shown in [50] that certain algebraic variants of multivariable Taylor towers for string links and braids contain all finite type invariants (and that's all they contain). It is known that such invariants separate braids [4]. It would be nice to reprove this separation result in the context of multivariable calculus. Similarly, if would be nice to reprove in manifold calculus setting that finite type invariants separate homotopy string links [17], although it is not yet clear that the Taylor multitower for those classifies finite type invariants (the main problem is that defining Bott-Taubes integrals for homotopy string links is tricky, if not impossible; see one of the other open problems below). In fact, more is known in the homotopy string links case: It was shown in [17] that Milnor invariants separate those, and it would thus be useful to see Milnor invariants in the Taylor multitower for homotopy string links.

(12) Work in [53] and [26] is related in the sense that both papers give results about the Vassiliev spectral sequence. Analogously, work in [35] and [3] should be related. Sakai constructs cohomology classes for more general spaces of embedding than knots, using configuration space integrals as in [53], while orthogonal spectral sequence in [3] is a generalization of Vassiliev's.

(13) It was shown in [26] that the Vassiliev cohomology spectral sequence [49] converging to long knots in $\mathbb{R}^n$, $n \geq 4$, collapses at E_1. It was also shown in [3] that a certain spectral sequence converging to the cohomology of $\mathrm{Emb}(M, V)$, where M is a manifold and V is a vector space, and arising from orthogonal calculus of functors, collapses rationally at E_1 (with some assumptions on dimensions). Specializing to $M = \mathbb{R}$ and $V = \mathbb{R}^n$, $n \geq 4$, shows that this orthogonal E_1 page and Vassiliev's E_1 page are quasi-isomorphic, up to regrading. Understanding the orthogonal spectral sequence better might shed more light on the structure of the rational cohomology of spaces of knots (with the ultimate goal of obtaining a closed form expression for this cohomology).

(14) Related to the previous problem, coformality of configuration spaces led the authors of [1] to deduce the rational collapse of the *homotopy* spectral sequence for long knots in $\mathbb{R}^n$, $n \geq 4$. The same result should be true more generally for $\mathrm{Emb}(M, V)$. This would allow us to deduce that $\mathrm{Emb}(M, V)$ is a rational homotopy functor of M and not just a rational homology functor, as was shown in [3].

(15) Are the cosimplicial spaces defined in [44] and [43] equivalent? They likely are (since they have the same totalization), but the first and last coface maps are different.

2. Other open problems

(1) What does the Taylor tower for knots (and Taylor multitowers for various link spaces) converge to? The conjecture is that it converges to the group completion of knots. Mostovoy's description of the delooping of long knots via short ropes [30] is almost certainly relevant here.

(2) Does the Taylor multitower for homotopy string links from [32] in $\mathbb{R}^n$, $n > 3$, converge, as it does for ordinary links? The convergence in the links case uses Goodwillie-Klein-Weiss estimates, but there is no such a result for homotopy links.

(3) How can one define (integral) finite type invariants of knots (or links in the multivariable case) in the Taylor tower, and not just in its algebraic variant [53]? Work of Budney, Conant, Scannell, and Sinha [10] is the first step in this direction. Conant [15] gives further evidence that the Taylor tower classifies finite type invariants. Adapting Koytcheff's construction [24] to the Taylor tower might also be useful.

(4) In theory, homology of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$ (long knots in $\mathbb{R}^n$), $n > 3$, is completely computable from its combinatorial characterization as essentially the Hochschild homology of the Poisson operad [26]. However, the computational complexity appears to grow exponentially and the computations have still been performed only in low degrees. What is needed is a better understanding of the structure of the Poisson operad. Turchin has already done a lot of work in this direction [48, 47, 46]. A way to get further might be to exploit formality and coformality at the same time. Namely, using [26] and [1], there are rational isomorphisms between the homology of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$ and the free graded algebra on the Yang-Baxter Lie algebra [20]. Both objects are accessible combinatorially and it would be good to examine both more closely, with an eye

toward obtaining a closed form description of $H_*(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n))$. There have also been some recent simplifications in the first differential of the spectral sequence giving rise to the description of homology of $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$ by Christine Pelatt.

(5) Because the authors of [26] could not prove directly that the Kontsevich operad $\mathcal{K}_n^\bullet$ was formal, the category of *fanic diagrams* was introduced and the proofs in [26, 1] became quite involved. However, it seems that if one were working with $\mathrm{Emb}(S^1, \mathbb{R}^n)$, i.e. usual (based) knots rather than long ones, the proofs would have been much simpler. Namely, spaces $\mathcal{F}_n(k)$ which constitute the Kontsevich operad do form a cosimplicial space $\mathcal{F}_n^\bullet$ if the requirement that two of the configuration points lie at $\pm\infty$ (which arises from punching holes in long knots; this is the first step in applying manifold calculus to knots) is dropped. To relate this to $\mathrm{Emb}(S^1, \mathbb{R}^n)$, it would be necessary to revisit Sinha's construction of $\mathcal{K}_n^\bullet$ from [44] and prove that the partial totalizations of $\mathcal{F}_n^\bullet$ are equivalent to the stages of the manifold calculus Taylor tower for $\mathrm{Emb}(S^1, \mathbb{R}^n)$. This seems well within reach.

(6) Budney shows that the classical long knots are a free little discs object over the prime long knots. If would be nice to establish an analog of this freeness result for knots in $\mathbb{R}^n$, $n > 3$. Bringing in Budney's work into the picture might also shed some light on the geometric interpretation of the generators of $H_*(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n))$.

(7) If one is to construct all (integral) finite type invariants using the Taylor tower for $\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$ and its various models (and not its algebraic variant), an important step might be to show that the map $\pi_0(\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)) \to \pi_0(T_r \,\mathrm{Emb}(\mathbb{R}, \mathbb{R}^3))$, where T_r is the rth stage of the Taylor tower, is a surjection for all r. This is precisely what is predicted by the convergence theorem of Goodwillie and Weiss [16]. Munson and I showed that this holds for small r via an unusual application of the Isotopy Extension Theorem to the cubical diagrams defining the stages T_r (the main trick was explained to us by Goodwillie). The hope is that these methods will generalize to all stages, but the problem seems to be quite difficult.

(8) It is known that Bott-Taubes configuration space integrals represent a universal finite type invariant of knots [45, 54] as well as links and braids [51]. It would be nice to show the same result for homotopy string links. The problem is that homotopy string links are usually defined as subspaces of the spaces of immersions, and not embeddings, in which case certain spaces built out of compactified configuration spaces which are central to the theory aren't necessarily manifolds with corners. Thus one cannot define Bott-Taubes integration for homotopy string links in the same way as for knots and links. One possible avenue of approach in resolving this issue would be to show that the resulting configuration spaces are *semi-algebraic bundles* over the space of homotopy string links and then do the integration in this setting, as described in [18]. But it's not clear that this would work either since homotopy string links can be given non-polynomially.

(9) Related to the previous item, it can be shown [50] that finite type invariants of homotopy string links do live in the (algebraic version of) the Taylor multitower [33]. Because of lack of Bott-Taubes integration, there could be more invariants in the multitower than the finite type ones. If Bott-Taubes integration cannot be extended to this context, is there some other way to show that the Taylor multitower only contains finite type invariants of homotopy string links? Further, exactly which of those are Milnor invariants (it is known that Milnor invariants of homotopy string links are finite type [4]) and how can they be identified?

(10) Even though Bott-Taubes configuration space integrals are mostly a technical tool from the point of view of homotopy theory, they are interesting in their own right because they represent a bridge between classical knot theory and physics (Chern-Simons theory is the birthplace of configuration space integrals) on one side and homotopy theory of embedding spaces on the other. In particular, the question of the vanishing of *anomalous faces* remains unsolved. Namely, it is now known whether Bott-Taubes integrals vanish when restricted to the face of a compactified configuration space characterized by all points coming together. D. Thurston's computations in low degrees suggest that this is so. Resolving this issue would also give a better relationship between Bott-Taubes and Kontsevich integration [22] (these are both ways of constructing a universal finite type knot invariant).

(11) Manifold calculus of functors might have an interesting connection to Khovanov homology [19]. To start, there is an easy observation that cubical diagrams involving similar things are central both to calculus and to Khovanov homology. More evidence comes from results of Przytycki [34] who relates Khovanov homology to Hochschild homology of links, and the latter (in some form) abounds in applications of manifold calculus to knot theory.

(12) Looking at [8, 35, 3], it seems that a lot of ideas in knot theory which come from calculus of functors, Bott-Taubes integration, and other places can be adapted to work for more general embedding spaces $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$. It would be nice to have a "punching-holes" or a (multi)cosimplicial model for this situation.

(13) Some results about $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ (modulo immersions) that use manifold calculus already exist. In [2], Arone and Turchin show that this space is equivalent to the space of truncated "weak" module maps from little j-cubes to little n-cubes and hence, if $n \geq j+3$ (so that the Taylor tower converges), $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ is the space of all such module maps. Turchin has further conjectured that $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ is $(j+1)$-fold loop space of the space of derived morphisms between j-cubes and n-cubes. The conjecture was verified by Turchin and Lambrechts for $j = 1$.

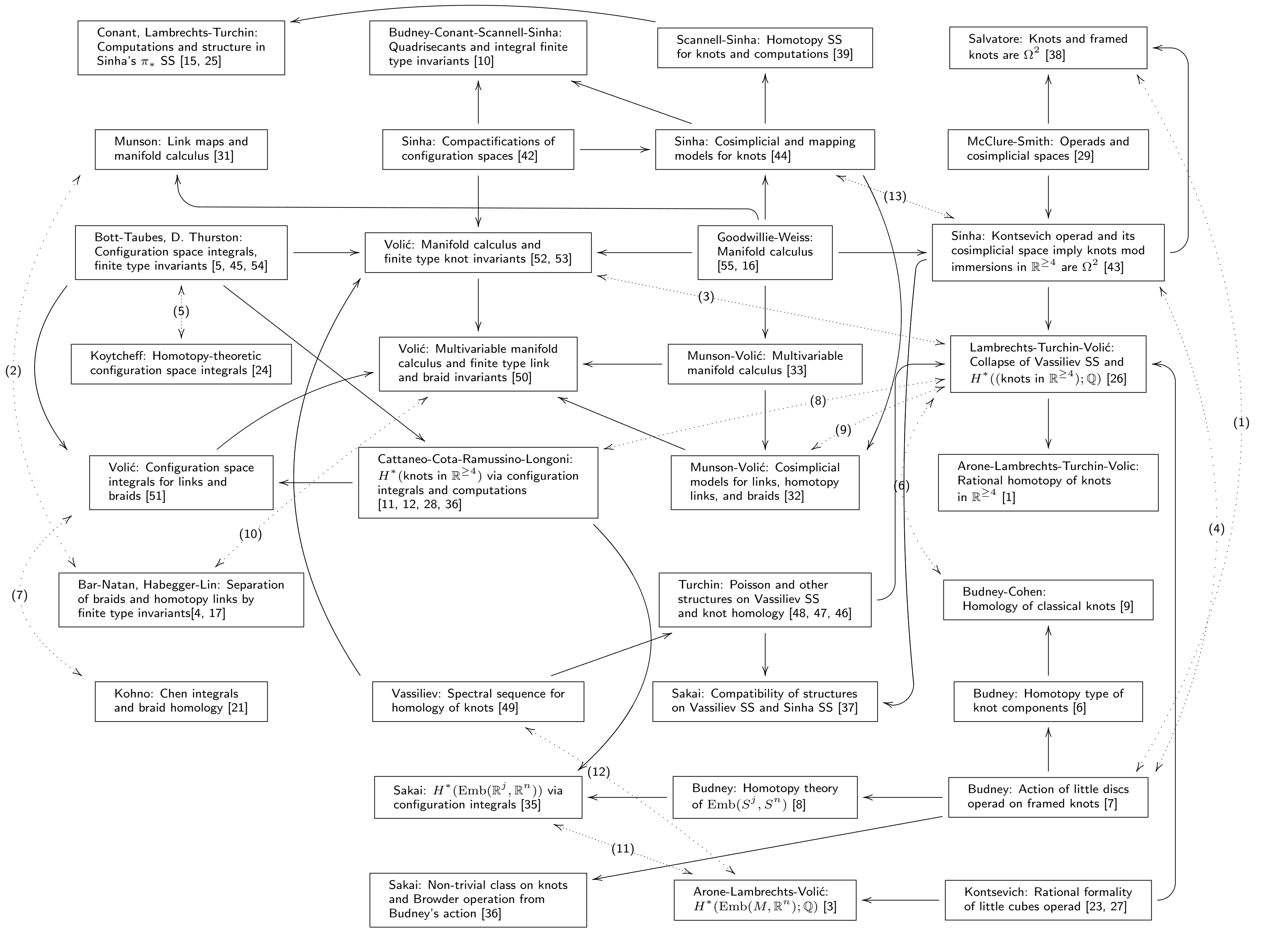
(14) Any sort of manifold calculus for $\mathrm{Emb}(\mathbb{R}^2, \mathbb{R}^4)$ is likely to connect to work of Schneiderman and Teichner [41, 40].

(15) If one studies $\mathrm{Emb}(\Sigma, \mathbb{R}^n)$, where Σ is a surface with some genus and some number of punctures, then the spaces in the Taylor tower would always be homotopy equivalent to wedges of circles. Maps between them in the (sub)cubical diagrams defining the stages of the Taylor tower would "double" circles, just like in the case of knots these maps "double" points. The doubling in case of knots was made precise via a cosimplicial model consisting of compactified configuration spaces. In case of $\mathrm{Emb}(\Sigma, \mathbb{R}^n)$, one would need to define some kind of a compactification of configurations of circles in $\mathbb{R}^n$. If this is doable, then a cosimplicial model for the Taylor tower for $\mathrm{Emb}(\Sigma, \mathbb{R}^n)$ would almost immediately follow. As was pointed out to me by Ben Cooper, this has the potential of connecting in interesting ways to $\mathrm{BDiff}(\Sigma)$ and Kontsevich's associative graph complex. In fact, the Taylor tower might give some kind of a filtration of $\mathrm{BDiff}(\Sigma)$ by graph complexity.

In the literature chart on the next page,

$A \longrightarrow B$ means results from A are used in B.
$A \dashleftarrow\dashrightarrow B$ means results from A and B should be related.

FIGURE 1. Literature on recent developments in knot and link theory using homotopy-theoretic methods such as manifold calculus, cosimplicial spaces and associated spectral sequences, and action of the little discs operad.



References

1. Greg Arone, Pascal Lambrechts, Victor Turchin, and Ismar Volić, *Coformality and rational homotopy groups of spaces of long knots*, Math. Res. Lett. **15** (2008), no. 1, 1–14.
2. Greg Arone and Victor Turchin, *On the rational homotopy type of high dimensional analogues of spaces of long knots*, in preparation.
3. Gregory Arone, Pascal Lambrechts, and Ismar Volić, *Calculus of functors, operad formality, and rational homology of embedding spaces*, Acta Math. **199** (2007), no. 2, 153–198.
4. Dror Bar-Natan, *Vassiliev homotopy string link invariants*, J. Knot Theory Ramifications **4** (1995), no. 1, 13–32.
5. Raoul Bott and Clifford Taubes, *On the self-linking of knots*, J. Math. Phys. **35** (1994), no. 10, 5247–5287, Topology and physics.
6. Ryan Budney, *Topology of spaces of knots in dimension* 3., Proc. Lon. Math. Soc., to appear.
7. ______, *Little cubes and long knots*, Topology **46** (2007), no. 1, 1–27.
8. ______, *A family of embedding spaces*, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 41–83.
9. Ryan Budney and Fred Cohen, *On the homology of the space of knots*, Geom. Topol. **13** (2009), no. 1, 99–139.
10. Ryan Budney, James Conant, Kevin P. Scannell, and Dev Sinha, *New perspectives on self-linking*, Adv. Math. **191** (2005), no. 1, 78–113.
11. Alberto S. Cattaneo, Paolo Cotta-Ramusino, and Riccardo Longoni, *Configuration spaces and Vassiliev classes in any dimension*, Algebr. Geom. Topol. **2** (2002), 949–1000 (electronic).
12. ______, *Algebraic structures on graph cohomology*, J. Knot Theory Ramifications **14** (2005), no. 5, 627–640.
13. F. R. Cohen and S. Gitler, *Loop spaces of configuration spaces, braid-like groups, and knots*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 59–78.
14. ______, *On loop spaces of configuration spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1705–1748 (electronic).
15. James Conant, *Homotopy approximations to the space of knots, Feynman diagrams, and a conjecture of Scannell and Sinha*, Amer. J. Math. **130** (2008), no. 2, 341–357.
16. Thomas G. Goodwillie and Michael Weiss, *Embeddings from the point of view of immersion theory II*, Geom. Topol. **3** (1999), 103–118 (electronic).
17. Nathan Habegger and Xiao-Song Lin, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. **3** (1990), no. 2, 389–419.
18. R. Hardt, P. Lambrechts, V. Turchin, and I. Volić, *Real homotopy theory of semi-algebraic sets*, submitted.
19. Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.
20. Toshitake Kohno, *Linear representations of braid groups and classical Yang-Baxter equations*, Braids (Santa Cruz, CA, 1986), Contemp. Math., vol. 78, Amer. Math. Soc., Providence, RI, 1988, pp. 339–363.
21. ______, *Loop spaces of configuration spaces and finite type invariants*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr., vol. 4, Geom. Topol. Publ., Coventry, 2002, pp. 143–160 (electronic).
22. Maxim Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97–121.
23. ______, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48** (1999), no. 1, 35–72, Moshé Flato (1937–1998).
24. Robin Koytcheff, *A homotopy-theoretic view of Bott-Taubes integrals and knot spaces*, Algebr. Geom. Topol. **9** (2009), no. 3, 1467–1501.
25. Pascal Lambrechts and Victor Turchin, *Homotopy graph-complex for configuration and knot spaces*, Trans. Amer. Math. Soc. **361** (2009), no. 1, 207–222.
26. Pascal Lambrechts, Victor Turchin, and Ismar Volić, *The rational homology of spaces of long knots in codimension* > 2, submitted, arXiv:math.AT/0703649.
27. Pascal Lambrechts and Ismar Volić, *Formality of the little* n*-discs operad*, in preparation, draft available at `http://palmer.wellesley.edu/~ivolic/pages/papers.html`.
28. Riccardo Longoni, *Nontrivial classes in* $H^*(\mathrm{Imb}(S^1, \mathbb{R}^n))$ *from nontrivalent graph cocycles*, Int. J. Geom. Methods Mod. Phys. **1** (2004), no. 5, 639–650.
29. James E. McClure and Jeffrey H. Smith, *Operads and cosimplicial objects: an introduction*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 133–171.
30. Jacob Mostovoy, *Short ropes and long knots*, Topology **41** (2002), no. 3, 435–450.
31. Brian A. Munson, *A manifold calculus approach to link maps and the linking number*, Algebr. Geom. Topol. **8** (2008), no. 4, 2323–2353.
32. Brian A. Munson and Ismar Volić, *Cosimplicial models for spaces of links*, submitted.
33. ______, *Multivariable manifold calculus of functors*, submitted.
34. Jozef Przytycki, *When the theories meet: Khovanov homology as hochschild homology of links*, arXiv: math.GT/0509334.
35. Keiichi Sakai, *Configuration space integrals for embedding spaces and the Haefliger invariant*, arXiv: 0811.3726.

36. ______, *Nontrivalent graph cocycle and cohomology of the long knot space*, Algebr. Geom. Topol. **8** (2008), no. 3, 1499–1522.
37. ______, *Poisson structures on the homology of the space of knots*, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 463–482.
38. Paolo Salvatore, *Knots, operads, and double loop spaces*, Int. Math. Res. Not. (2006), Art. ID 13628, 22. MR MR2276349 (2007j:57012)
39. Kevin P. Scannell and Dev P. Sinha, *A one-dimensional embedding complex*, J. Pure Appl. Algebra **170** (2002), no. 1, 93–107.
40. Rob Schneiderman and Peter Teichner, *Higher order intersection numbers of 2-spheres in 4-manifolds*, Algebr. Geom. Topol. **1** (2001), 1–29 (electronic).
41. ______, *Whitney towers and the Kontsevich integral*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 101–134 (electronic).
42. Dev P. Sinha, *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. (N.S.) **10** (2004), no. 3, 391–428.
43. ______, *Operads and knot spaces*, J. Amer. Math. Soc. **19** (2006), no. 2, 461–486 (electronic).
44. ______, *The topology of spaces of knots: cosimplicial models*, Amer. J. Math. **131** (2009), no. 4, 945–980.
45. Dylan Thurston, *Integral expressions for the Vassiliev knot invariants*, arXiv:math.AT/0701350.
46. V. Tourtchine, *On the other side of the bialgebra of chord diagrams*, J. Knot Theory Ramifications **16** (2007), no. 5, 575–629.
47. Victor Tourtchine, *What is one-term relation for higher homology of long knots*, Mosc. Math. J. **6** (2006), no. 1, 169–194, 223.
48. V. Tourtchine (Turchin), *On the homology of the spaces of long knots*, Advances in topological quantum field theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 179, Kluwer Acad. Publ., Dordrecht, 2004, pp. 23–52.
49. V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 23–69.
50. Ismar Volić, *Finite type invariants of links and braids via manifold calculus*, in preparation.
51. Ismar Volić, *On the cohomology of spaces of links and braids via configuration space integrals*, to appear in Sarajevo J. Math.
52. Ismar Volić, *Configuration space integrals and Taylor towers for spaces of knots*, Topology Appl. **153** (2006), no. 15, 2893–2904.
53. ______, *Finite type knot invariants and the calculus of functors*, Compos. Math. **142** (2006), no. 1, 222–250.
54. ______, *A survey of Bott-Taubes integration*, J. Knot Theory Ramifications **16** (2007), no. 1, 1–42.
55. Michael Weiss, *Embeddings from the point of view of immersion theory I*, Geom. Topol. **3** (1999), 67–101 (electronic).