CALCULUS OF FUNCTORS AND LINK INVARIANTS

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<u>Goal</u>: study invariants (elements of H^0) of various knot/link spaces using manifold (embedding) calculus of functors.

Let

$\mathcal{K}^d = \{\text{embeddings/immersions } \mathbb{R} \to \mathbb{R}^d\}$
= <u>space of long knots</u>;

$\mathcal{L}k_n^d = \{\text{embeddings/immersions } \coprod_n \mathbb{R} \to \mathbb{R}^d\}$
= <u>space of long (string) links</u>;

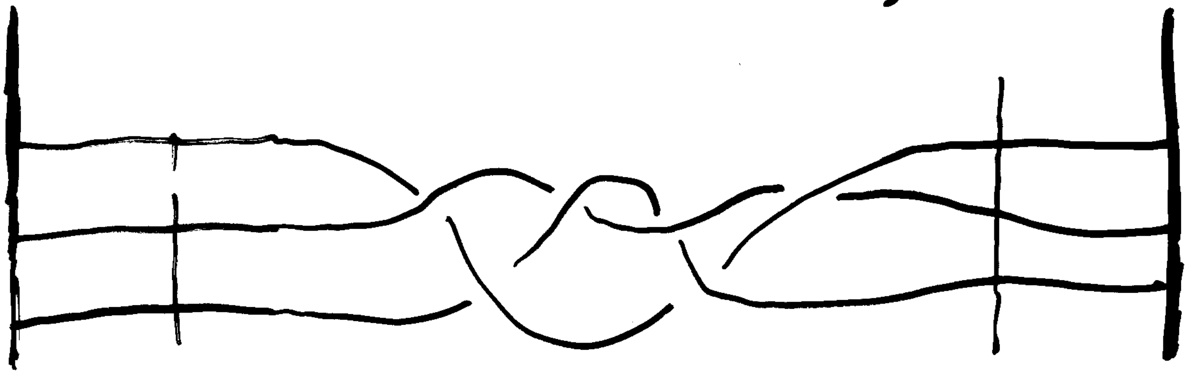
$\mathcal{HL}k_n^d = \{\text{link maps } \coprod_n \mathbb{R} \to \mathbb{R}^d\}$
= <u>space of homotopy string links</u>;

$Br_n^d = \{\text{embeddings/immersions } \coprod_n \mathbb{R} \to \mathbb{R}^d \text{ with positive derivative}\}$
= <u>space of braids</u>;

such that behavior is fixed outside a compact set.

(embeddings/immersions means take hofiber(embeddings $\hookrightarrow$ immersions))

$\in \mathcal{HL}k_3^d$



$\in \mathcal{B}r_3^d \in \mathcal{HL}k_3^d \subset \mathcal{L}k_3^d$

$H_0(X)$ = {connected components of X}
= {isotopy or homotopy classes}
= {knot/link types}

$H^0(X)$ = {functions constant on connected components}
= <u>invariants</u>

This is interesting for $d=3$.

Goodwillie-Weiss manifold (embedding) calculus

M, N manifolds

$\mathcal{O}(M)$ = category of open subsets of M w/ inclusions
$\mathcal{O}_k(M)$ = subcategory of $\mathcal{O}(M)$ of open subsets diffeomorphic to at most k disjoint balls

$$F : \mathcal{O}(M)^{op} \longrightarrow \begin{matrix}\text{Top} \\ \text{Spectra}\end{matrix}$$

Get "Taylor tower" of functors / fibrations

$$F(-) \longrightarrow \left(T_\infty F(-) \to \cdots \to T_k F(-) \to \;--\; \to T_0 F(-) \right)$$

where

$$\boxed{T_k F(V) = \underset{U \in \mathcal{O}_k(V)}{\operatorname{holim}} F(U)}$$

<u>Thm (Goodwillie-Klein-Weiss):</u>

For $F = \mathrm{Emb}(M, N)$, tower converges if $\dim M + 2 < \dim N$.

Apply this calculus to

$F = \mathcal{K}^d, \mathcal{L}h_n^d, \mathcal{H}\mathcal{L}h_n^d$, and $\mathcal{B}r_n^d$,

mostly for $d=3$ — no convergence result, but still get lots of information about H^0 from tower. Also, construction of T_k for these functors simplifies:

Special case $\mathcal{K}^d$:

Let $I_1, \dots, I_k$ be disjoint subintervals of $\mathbb{R}$.

For $\emptyset \neq S \subseteq \{1, \dots, k\}$, let

$$\mathcal{K}_S^d = \mathrm{Emb}\left(\mathbb{R} \setminus \bigcup_{i \in S} I_i, \mathbb{R}^d\right)$$

(space of "punctured knots")

Have restriction maps

$$\mathcal{K}_S^d \longrightarrow \mathcal{K}^d_{S \cup \{i\}}$$

(restrict a punctured knot to a knot with one more puncture)

Get (sub)cubical diagram of $\mathcal{K}_S^d$'s.
Then

$$T_k \mathcal{K}^d = \operatorname*{holim}_{\emptyset \neq S \subseteq \{1, \dots, k\}} \mathcal{K}_S^d$$

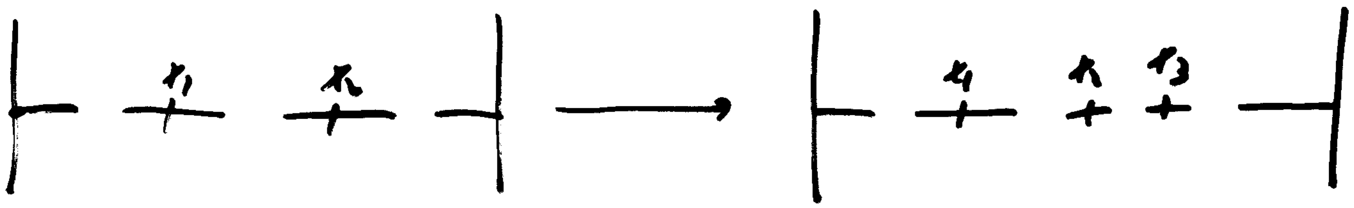
<u>Easy to see</u>: For $k > 3$, $\mathcal{K}^d$ is the actual pullback of the subcubical diagram.

<u>Note</u>:

$$\mathcal{K}_S^d \simeq C(|S|-1, \mathbb{R}^d)$$

= configuration space of $|S|-1$ points in $\mathbb{R}^d$.

Restriction maps "add a point":



$$C(2, \mathbb{R}^d) \longrightarrow C(3, \mathbb{R}^d)$$

To make this precise, have

Cosimplicial model for Taylor tower for $\mathcal{K}^d$

Let

$$C\langle p, \mathbb{R}^d\rangle \overset{\text{drop } \mathbb{R}^d}{=} C\langle p\rangle = \text{(a quotient of) Fulton - MacPherson compactification of } C(p, \mathbb{R}^d)$$

This is :

- homotopy equivalent to $C(p)$
- points allowed to come together but directions of approach are recorded
- codimension 1 faces characterised by points coming together at same time (important for Stokes' Theorem arguments)
- $\{C\langle p\rangle\}_{p=0}^{\infty}$ is the Kontsevich operad

Define cosimplicial space

$$K^\bullet = \left(C\langle 0\rangle \underset{s^i}{\overset{d^i}{\rightrightarrows}} C\langle 1\rangle \underset{s^i}{\overset{d^i}{\rightrightarrows}} C\langle 2\rangle \rightrightarrows \cdots \right)$$

where d^i = doubling
s^i = forgetting

<u>Thm</u> (<u>Sinha</u>): $\mathrm{Tot}^k K^\bullet \simeq T_k \mathcal{K}^d, \ d \geq 2$

Now can use Bousfield-Kan H_* and π_* spectral sequences for $K^\bullet$

Thm (Arone, Lambrechts, Turchin, V):

These spectral sequences collapse rationally at E^2 for $d>3$.

(and because Taylor tower converges for $d>3$, get complete description of rational homotopy type of K^d, $d>3$.)

Here I'm interested in H^* spectral sequence for $d=3$ (more later).

Two-variable manifold calculus
(n-variable similar)

If $N = P \amalg Q$, can apply 2-variable calculus for contravariant functors F on $\mathcal{O}(P) \times \mathcal{O}(Q)$ (rather than $\mathcal{O}(P \amalg Q)$).

Define bitower

$$
F(-,-) \longrightarrow
\begin{tikzcd}
T_{\infty,\infty} F(-,-) \arrow[d] & \cdots \arrow[l] & \cdots & \cdots \arrow[l] & T_{n,\infty} F(-,-) \arrow[l] \arrow[d] \\
\vdots \arrow[d] & & & & \vdots \arrow[d] \\
T_{0,1} F(-,-) \arrow[d] & T_{1,1} F(-,-) \arrow[l] & \cdots \arrow[l] & & T_{n,1} F(-,-) \arrow[ll] \arrow[d] \\
T_{0,0} F(-,-) & T_{1,0} F(-,-) \arrow[l] & \cdots \arrow[l] & & T_{n,0} F(-,-) \arrow[ll]
\end{tikzcd}
$$

where

$$
(T_{k_1,k_2} F)(V_1, V_2) = \operatorname*{holim}_{(U_1,U_2) \in \mathcal{O}_{k_1}(V_1) \times \mathcal{O}_{k_2}(V_2)} F(U_1, U_2)
$$

Not hard to see:

- $T_k F = \operatorname*{holim}_{k_1 + k_2 = k} T_{k_1,k_2} F$
- Have same G-K-W convergence result

Special case $\mathcal{L}k_2$ $(d \geq 2)$:

Let $I_1, \dots, I_{k_1}$ be disjoint in $\mathbb{R}$
$J_1, \dots, J_{k_2}$ be disjoint in $\mathbb{R}$

Then

$$T_{k_1,k_2}\mathcal{L}k_2 = \underset{\substack{\emptyset \neq S_1 \subseteq \{1,\dots,k_1\} \\ \emptyset \neq S_2 \subseteq \{1,\dots,k_2\}}}{\mathrm{holim}} \mathrm{Emb}\Big(\mathbb{R} \setminus \bigcup_{i \in S_1} I_i \amalg \mathbb{R} \setminus \bigcup_{j \in S_2} J_j, \mathbb{R}^d\Big)$$

$= \{$punctured links of 2 components$\}$

In analogy with knots, get bicosimplicial space

$$LK_2^{\bullet,\bullet} = \left(\begin{array}{ccccccc} \vdots & & & & & & \\ \Uparrow\!\!\Uparrow \downdownarrows & & & & & & \\ \langle\langle 0+2 \rangle\rangle & \rightleftarrows & \langle\langle 1+2 \rangle\rangle & \underset{s^i}{\overset{d^i}{\rightleftarrows}} & \cdots & & \\ \Uparrow\!\!\Uparrow \downdownarrows & & \Uparrow\!\!\Uparrow \downdownarrows & & & & \\ \langle\langle 0+1 \rangle\rangle & \rightleftarrows & \langle\langle 1+1 \rangle\rangle & \rightleftarrows & \langle\langle 2+1 \rangle\rangle & \rightleftarrows & \cdots \\ \Uparrow \downarrow & & \Uparrow \downarrow & & \Uparrow \downarrow & & \\ \langle\langle 0 \rangle\rangle & \rightleftarrows & \langle\langle 1+0 \rangle\rangle & \rightleftarrows & \langle\langle 2+0 \rangle\rangle & \rightleftarrows & \cdots \end{array} \right)$$

Again d^i = doubling, s^i = forgetting

Prop: $\mathrm{Tot}^{k_1,k_2} LK_2^{\bullet,\bullet} \simeq T_{k_1,k_2}\mathcal{L}k_2$

Now consider the diagonal cosimplicial space

$$LK^{\bullet,\bullet}_{2,diag} = \Big(C\langle 0+0\rangle \rightleftarrows C\langle 1+1\rangle \underset{s^i = s^i \circ s^i}{\overset{d^i = d^i\circ d^i}{\rightleftarrows}} C\langle 2+2\rangle \cdots \Big)$$

Then

$$\mathrm{Tot}\, LK_2^{\bullet,\bullet} \simeq \mathrm{Tot}\, LK^{\bullet,\bullet}_{2,diag}$$

Have Bousfield-Kan H^* spectral sequence for $LK^{\bullet,\bullet}_{2,diag}$ with

$$E_1^{-p,q} = H^q(C\langle 2p\rangle)$$

which converges to

$$\begin{cases} H^*(\mathrm{Tot}\, \overset{\text{cochains}}{C^*} LK^{\bullet,\bullet}_{2,diag}) = H^*(\mathrm{Tot}\, C^* LK_2^{\bullet,\bullet}), & d = 3 \\ H^*(\mathrm{Tot}\, LK^{\bullet,\bullet}_{2,diag}) = H^*(\mathcal{Y}k_2), & d > 3 \end{cases}$$

The difference comes from

This is what ss converges to

$$H^*(\mathrm{Tot}\, C^* LK_2^{\bullet,\bullet})$$

$$\downarrow$$

Have strong convergence, and hence equivalence here, for $d > 3$

$$H^*(\mathrm{Tot}\ LK_2^{\bullet,\bullet})$$

$$\|$$

Bicosimplicial space models bitower for all d.

$$H^*(T_{\infty,\infty}\, \mathcal{L}h_2)$$

$$\|$$

Single-variable and two-variable tower converge to same thing for all d.

$$H^*(T_\infty\, \mathcal{L}h_2)$$

$$\downarrow$$

Have G-K-W convergence of tower, and hence equivalence here for $d > 3$.

$$H^*(\mathcal{L}h_2)$$

<u>For $\mathcal{HL}_{h_2}$</u>: Again have bicosimplicial model for the bitower of punctured homotopy limits, except (h_1, h_2) entry is

$$C(h_1, h_2) = \{ (x_1, \dots, x_{h_1}, y_1, \dots, y_{h_2}) \in (\mathbb{R}^d)^{h_1 + h_2} \\ : x_i \neq y_j \}$$

$$= \text{"partial configuration space"}$$

(no need to compactify)

Again have

$$\mathrm{Tot}\, HLK_{2,\mathrm{diag}}^{\bullet,\bullet} = \mathrm{Tot}\, HLK_2^{\bullet,\bullet} \simeq T_{\infty,\infty} \mathcal{HL}_{h_2}$$

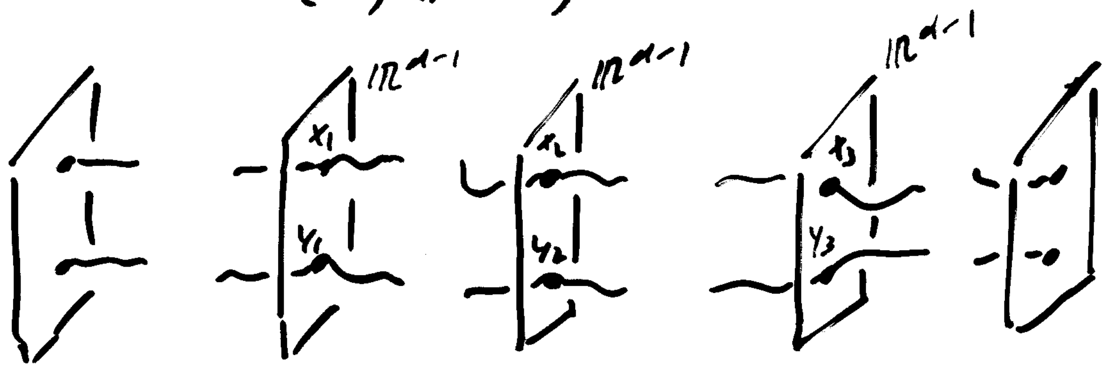
and H^* s.s. with

$$E_1^{-p,q} = H^q(C(p,p))$$

$H^*(C(p,p))$ is generated by subset of generators of $H^*(C(2p))$ with same relations (more on this later).

<u>For Br_2:</u> Everything same, except k^{th} space in $BR^{\bullet,\bullet}_{2,diag}$ is (after reparametrizing)

$$C(2, \mathbb{R}^{d-1})^{k-1}$$



So get cosimplicial model for loop space and hence

$$\mathrm{Tot}\ BR^{\bullet,\bullet}_{2,diag} = \Omega\, C(2, \mathbb{R}^{d-1})$$

(as should be the case!)

Generalization to n-component links straightforward: Get

- n-dim'l Taylor towers for $\mathcal{L}_n$, $H\mathcal{L}_n$, Br_n ;
- n-cosimplicial models for the towers ;
- diagonal cosimplicial spaces $\mathcal{L}_{n,diag}^{\bullet,\bullet,\dots,\bullet}$, $H\mathcal{L}_{n,diag}^{\bullet,\bullet,\dots,\bullet}$, $Br_{n,diag}^{\bullet,\bullet,\dots,\bullet}$;
- B-K H^* s.s.'s for these cosimpl. spaces.

<u>Thm (Munson, V)</u>: These s.s's collapse at E_2 on the diagonal (H^0 part) for $d=3$.

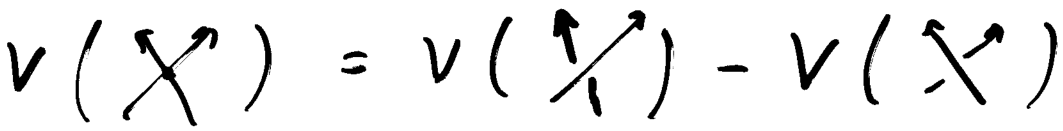
(True for $K^\bullet$ as well)

What's this E_2?

Finite type invariants

$d=3$.
Let $F = \mathcal{K}, \mathcal{L}k_n, \mathcal{HL}k_n$, or $\mathcal{B}r_n$ as usual.

An invariant $V \in H^0(F)$ can be extended to <u>singular</u> <u>knots/links</u> (finite number of transverse self-intersections) via <u>Vassiliev skein relation</u>



For $\mathcal{HL}k_n$, two strands are required to come from different [redacted] components (and they necessarily will for $\mathcal{B}r_n$).

So m-singular knot/link $\rightsquigarrow$ 2^m knots/links

<u>Def</u>: V is finite (Vassiliev) type m if it vanishes on $(m+1)$-singular knots/links.

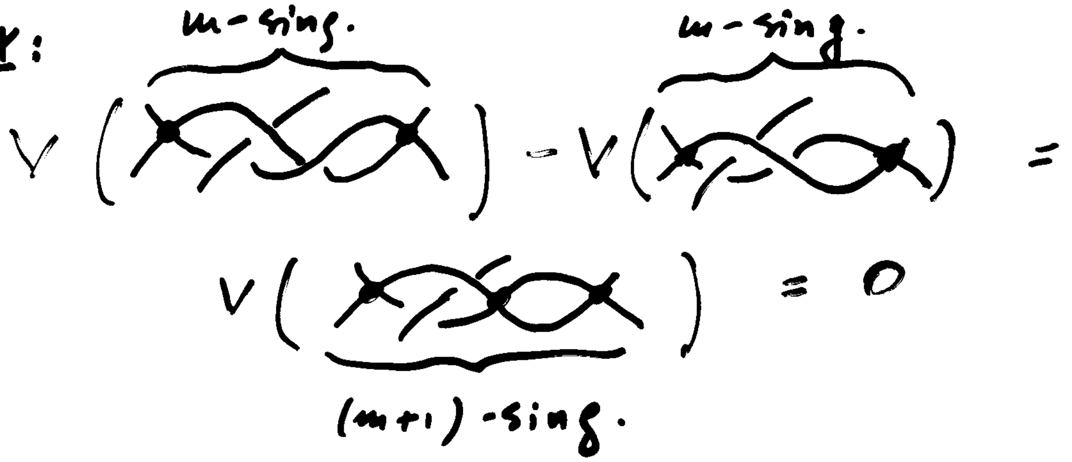
<u>Ex</u>:
- polynomial invariants
- Milnor invariants

- Finite type invariants are conjectured to separate knots/links.

let $\mathcal{V}_m = \{\text{type } m \text{ invariants}\}$

Note: Value of $V \in \mathcal{V}_m$ on an m-singular knot or link only depends on the placement of singularities and not on embedding.

ex:



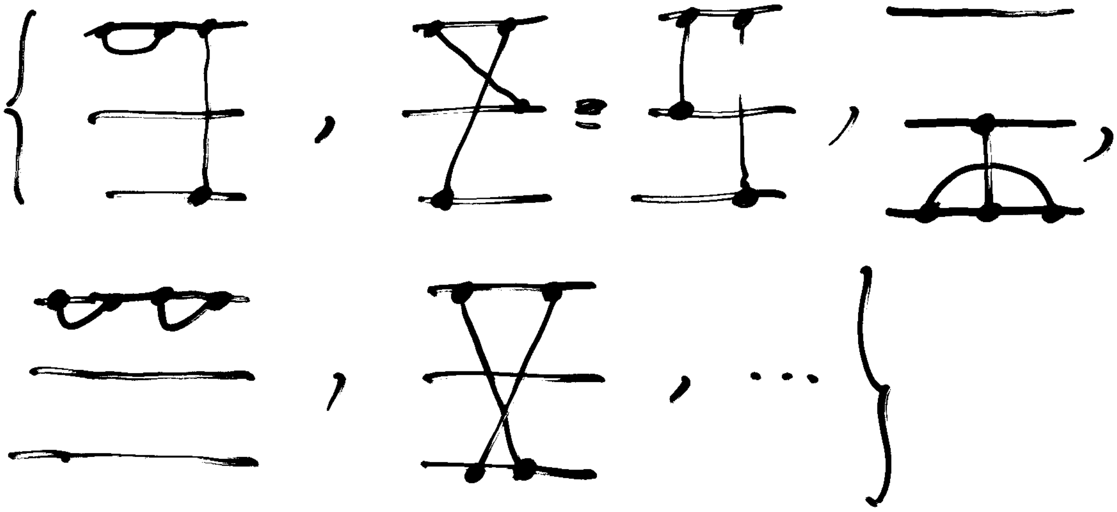
So V doesn't see knotting in m-singular knots / links.

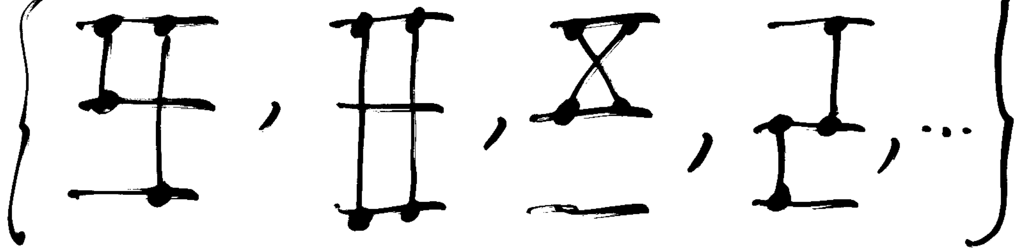
Placement of singularities can be encoded by **chord diagrams**.

let

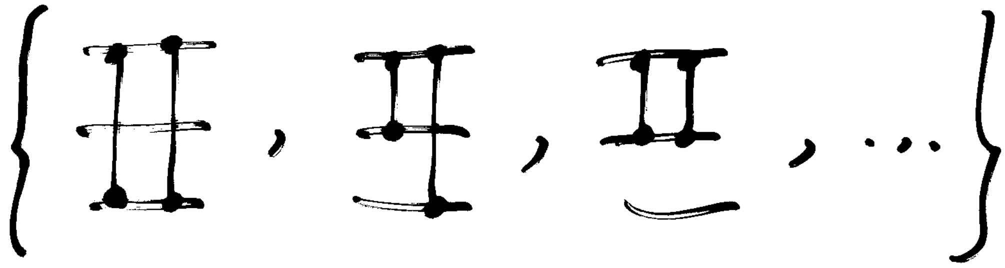
$$CD_m(F) = \left\{ \begin{array}{l} \text{chord diagrams associated} \\ \text{to } F \text{ with } m \text{ chords} \end{array} \right\}$$

ex:

$CD_2(\mathcal{L}h_3) =$ 

$CD_2(\mathcal{HL}h_3) =$ 

(no chords on single strand)

$CD_2(Br_3) =$ 

(parallel chords only)

Chord diagrams should be thought of as prescriptions for creating singularities.

So for $D \in CD_m(F)$, have map

$$\mathcal{V}_m \longrightarrow \{ f : \mathbb{Q}[CD_m(F)] \longrightarrow \mathbb{Q} \}$$

$$f(D) = V(L_D),$$

where L_D is any m-singular knot/link with singularities as prescribed by D.

Easy to see:
- Kernel is $\mathcal{V}_{m-1}$ (by definition)
- Image is at most those f which vanish on four-term ($4T$) relation.

Let

$$\mathcal{W}_m(F) = \{ f : \mathbb{Q}[CD_m F] / 4T \longrightarrow \mathbb{Q} \}$$

So have map

$$\mathcal{V}_m / \mathcal{V}_{m-1} \longrightarrow \mathcal{W}_m$$

Thm (Kontsevich): This is an isomorphism

Map back is the Kontsevich Integral. Alternatively, can use Bott-Taubes configuration space integrals.

What does this have to do with Taylor towers?

Recall that Bousfield-Kan H^* ss's had H^*(configuration spaces) in E_1.

This H^* can be represented by <u>chord diagrams</u>.

<u>ex</u>: For $\mathcal{K}$, have

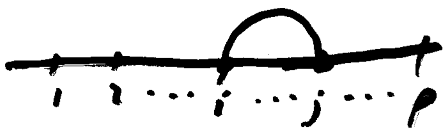
$$E_1^{-p,q} = H^q(C(p,\mathbb{R}^3))$$

and H^* is generated by classes

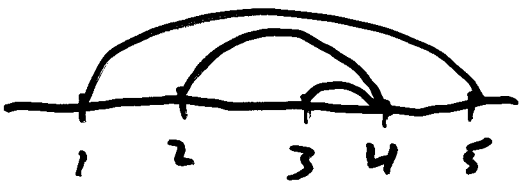
$$\alpha_{ij}, \quad 1 \le i < j \le p, \quad \deg \alpha_{ij} = 2$$

(modulo relations)

But can represent α_{ij} by



So, e.g.



$\longleftrightarrow \alpha_{15}\alpha_{23}\alpha_{34} \in H^6(C(5))$

Looking at the diagonal $E_1^{-p,p}$, we get (after normalizing $K^\bullet$),

$$E_1^{-2m,2m}(K^\bullet) \cong CD_m(X)$$

(and odd terms are 0).

Same for $F = \mathcal{L}k_n$, $H\mathcal{L}k_n$, or $\mathcal{B}r_n$.

Direct computation of d_1 gives

Prop: $E_2^{-2m,2m}(F) = W_m$

So we see finite type theory is the Taylor towers (or their cosimplicial replacements). But that's exactly what we see at E_∞ according to the main theorem. A restatement is:

<u>Thm</u>: All maps in following commutative diagram are isomorphisms:

$$
\begin{tikzcd}
H^0(T^*_{2m,2m,\ldots,2m}F)\big/H^0(T^*_{2m-1,\ldots,2m-1}F) \cong E_\infty^{-2m,2m}(F) \arrow[r, hook] \arrow[d, hook] & V_m/V_{m-1} \arrow[d] \arrow[d, bend left=60, "\text{easy map from before}"] \\
E_2^{-2m,2m} \arrow[r, "\cong \text{ by previous page}"'] & W_m \arrow[ul, dashed]
\end{tikzcd}
$$

Key ingredient: Extension of Bott-Taubes integrals to Taylor towers

<u>Corollaries</u>:

- $H^0(T^*_{2m,2m,\ldots,2m}F) \cong V_m$
- Recover Kontsevich's theorem

So $\boxed{\text{(Multivariable) Taylor towers classify finite type invariants of knots and links.}}$

To do:

① Reprove, in this setting, that finite type invariants separate braids (Kohno, Bar-Natan) and homotopy string links (Habegger-Lin).

② See if this helps in proving the same result for knots (and links). This is a question about whether there is a surjection

$$H^0(\mathcal{K}) \twoheadleftarrow H^0(T_\infty^* \mathcal{K}) = \text{finite type invariants}$$

③ Show rational collapse of Bousfield-Kan H^* s.s. for $\mathcal{L}_n$ and $\mathcal{H}\mathcal{L}_n$ for $d>3$ (analog of Lambrechts-Turchin-V. result for $\mathcal{K}^d$).

Generalizations of Milnor Invariants

A link map

$$\coprod_{i=1}^{n} f_i \; : \; \coprod_{i=1}^{n} S^{p_i} \longrightarrow \mathbb{R}^d$$

determines a map

$$f = \prod_{i=1}^{n} f_i \; : \; \prod_{i=1}^{n} S^{p_i} \longrightarrow C(n, \mathbb{R}^d)$$

<u>Koschorke</u>: $[f] \in [\, \prod S^{p_i}, C(n, \mathbb{R}^d)]$ is an invariant of $\coprod f_i$.

Further, suppose $p_i = 1$ and link is <u>almost trivial</u>, i.e. $\forall j \in \{1, \ldots, n\}$ the map

$$\hat{f}_j : \prod_{i \neq j} S^1 \longrightarrow C(n-1, \mathbb{R}^3)$$

is null-homotopic (removing any link component gives unlink), then

$[f]$ is Milnor invariant (of length n).

Using this, have

<u>Thm (Koschorke)</u>: Almost trivial homotopy string link $\amalg f_i : \amalg S^1 \to \mathbb{R}^3$ is homotopy trivial iff $[f] = 0$.

<u>Conjecture</u>: Homotopy class of every link map $\amalg f_i$ is uniquely determined by $[f]$.

<u>Goal</u>: Show this using manifold calculus by reinterpreting

$$f : \prod S^1 \to C(n, \mathbb{R}^3)$$

as living in the n^{th} layer (fiber of $T_n \to T_{n-1}$) of Taylor tower for $\mathcal{L}h_n^3$.

<u>Idea</u>: Define <u>relative</u> Milnor invariants

<u>First step</u>: Suppose f and g are n-component homotopy links. Suppose relative Milnor invariants of length $< n$ of f and g vanish. Then

f h'topic to g $\iff$ relative Milnor invariants of length n are same for f and g.

Review of Milnor invariants

$F(m)$ = free group on m generators x_i

For G a group, let

$G_i = [G, G_{i-1}]$ = i^{th} commutator in lower central series

Recall Magnus expansion:

Let $P = \mathbb{Z}[[x_1, \ldots, x_m]]$
Have injective homomorphism

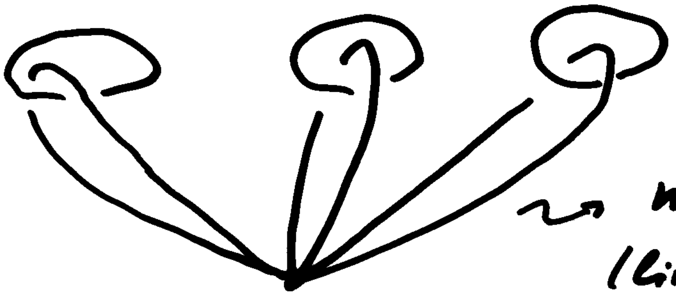
$$\begin{aligned} \sigma : \mathbb{Z}F(m) &\longrightarrow P \\ x_i &\longmapsto 1 + x_i \\ x_i^{-1} &\longmapsto 1 - x_i + x_i^2 - x_i^3 + \cdots \end{aligned}$$

So for $y \in F(m)$,

$$\sigma(y) = 1 + \sum \mu(l_1, \ldots, l_s)\, x_{l_1} \cdots x_{l_s}$$

$$(y \in F_h \iff \mu(l_1, \ldots, l_s) = 0 \text{ for } s \leq h-1)$$

Let $L \in \mathcal{L}h_m^3$. Then have meridians and longitudes of L:



$\rightsquigarrow$ meridians (linking # w/ components $= 1$)

Longitudes are displaced copies of link components.

Meridians determine map

$$\vee S^1 \longrightarrow \mathbb{R}^3 \setminus L$$

and induced map

$$(*) \qquad F(m) \longrightarrow \pi_1(\mathbb{R}^3 \setminus L) = \pi_L$$

Have presentation (Milnor)

$$\pi_L / [\pi_L, (\pi_L)_k] = \langle x_1, \dots, x_m \mid [x_i, l_i], F_{k+1} \rangle$$

where

- x_i is image of ith generator under ⊛
- l_i is word in x_i's representing the image of ith longitude in $\pi_L / [\pi_L, (\pi_L)_k]$
- F_{k+1} is subgroup of π_L generated by all $(k+1)$-fold commutators in x_i's.

Then <u>Milnor invariant</u> $\bar{\mu}_L(l_1, \dots, l_k, j)$ is the coefficient of $\kappa_{l_1} \cdots \kappa_{l_k}$ in the Magnus expansion $\sigma(l_j)$ of l_j.

This is the invariant of length $k+1$.