# Calculus of functors and link invariants

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# Plan of talk

**Goal:** Use Goodwillie-Weiss manifold calculus of functors to study invariants of various knot and link spaces.

**Outline**:

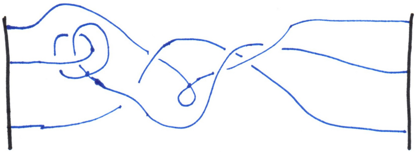
1. Goodwillie-Weiss single-variable manifold calculus, which leads to
2. Cosimplicial model for the space of knots and associated spectral sequences
3. Multivariable manifold calculus, which leads to
4. Multi-cosimplicial model for links, homotopy links, and braids, and associated spectral sequences
5. Connection to finite type knot/link invariants.

# Definitions of the knot and link spaces

$$\begin{aligned}
\mathcal{K}^d &= \{\text{embeddings } \ \mathbb{R} \hookrightarrow \mathbb{R}^d\} \\
&= \textit{space of long knots} \\
\mathcal{LK}^d_n &= \{\text{embeddings } \ \sqcup_n \mathbb{R} \hookrightarrow \mathbb{R}^d\} \\
&= \textit{space of long (string) links} \\
\mathcal{HLK}^d_n &= \{\text{link maps } \ \sqcup_n \mathbb{R} \hookrightarrow \mathbb{R}^d\} \\
&= \textit{space of homotopy long (string) links} \\
\mathcal{BR}^d_n &= \{\text{embeddings with positive derivative } \ \sqcup_n \mathbb{R} \hookrightarrow \mathbb{R}^d\} \\
&= \textit{space of long braids}
\end{aligned}$$

- All maps are standard outside a compact set;
- We actually take the homotopy fiber of the inclusion embeddings ↪ immersions;
- A *link map* is a smooth map with images of the copies of $\mathbb{R}$ disjoint.

# Examples



$\in \mathcal{HLK}_3^d$

$\in \mathcal{BR}_3^d \subset \mathcal{HLK}_3^d \subset \mathcal{LK}_3^d$

For $X = \mathcal{K}^3,\ \mathcal{LK}^3_n,\ \mathcal{HLK}^3_n$, or $\mathcal{BR}^3_n$, we care about

- $H_0(X) = \{\text{knot/link types}\}$
- $H^0(X) = \{\text{knot/link invariants}\}$

# Goodwillie-Weiss manifold (embedding) calculus

For $M$ a manifold, let

$$\mathcal{O}(M) = \text{category of open subsets of } M \text{ with inclusions.}$$

Given a functor

$$F\colon \mathcal{O}(M)^{op} \longrightarrow \mathcal{C}$$

where $\mathcal{C}$ is a model category, one has a "Taylor tower" of approximating functors/fibrations

$$F(-) \longrightarrow \big(T_\infty F(-) \to \cdots \to T_k F(-) \to \cdots \to T_0 F(-)\big)$$

**Theorem (Goodwillie-Klein-Weiss)**

*For the embedding functor* $\mathrm{Emb}(M, N)$, *where* $M, N$ *are manifolds, the Taylor tower converges if* $dim(M) + 2 < dim(N)$.

Note that this says nothing about knots and links in $\mathbb{R}^3$. Nevertheless, the tower still contains lots of information about classical knots.

# Construction of $T_kF$ for $F = \mathcal{K}^d$

Let $I_1, ..., I_k$ be disjoint subintervals of $\mathbb{R}$. For $\emptyset \neq S \subseteq \{1, .., k\}$, let

$$\mathcal{K}^d_S = \mathrm{Emb}(\mathbb{R} \setminus \bigcup_{i \in S} I_i,\ \mathbb{R}^d) = \text{space of ``punctured knots''}$$

Have restriction maps $\mathcal{K}^d_S \to \mathcal{K}^d_{S \cup \{i\}}$ which form a (sub)cubical diagram of punctured knots. Then

**Definition**

$$T_k\mathcal{K}^d = \operatorname*{holim}_{\emptyset \neq S \subseteq \{1,..,k\}} \mathcal{K}^d_S.$$

Easy to see: For $k > 3$, $\mathcal{K}^d$ is the actual pullback of the subcubical diagram.

# Cosimplicial model for $T_k\mathcal{K}^d$

Note that

$$\mathcal{K}_S^d = Conf(|S|-1, \mathbb{R}^d) = \text{configuration space of } |S|-1 \text{ points in } \mathbb{R}^d$$

and that restriction maps “add a point”. To make this precise, define cosimplicial space

$$\begin{tikzcd} K^\bullet = \big(\big(Conf(0,\mathbb{R}^d) \arrow[r, shift left=2] \arrow[r, shift right=2] & Conf(1,\mathbb{R}^d) \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & Conf(2,\mathbb{R}^d) \cdots \big), \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

where the cofaces $d^i$ are doubling (diagonal) maps and the codegeneracies $s^i$ are forgetting maps. Let $\mathrm{Tot}^k K^\bullet$ be the $k$th partial totalization of $K^\bullet$.

**Theorem (Sinha)**

*For $d \geq 2$, $\mathrm{Tot}^k K^\bullet \simeq T_k\mathcal{K}^d$.*

Now have Bousfield-Kan $H^*$ and $\pi_*$ spectral sequences for $K^\bullet$. For $d > 3$, these spectral sequences collapse rationally at the second page (Lambrechts-Turchin-V and Arone-Lambrechts-Turchin-V). But in this talk, we are interested in $H^0$ for $d = 3$ (invariants).

# Two-variable manifold calculus ($n$-variable similar)

If $M = P \coprod Q$, can apply two-variable calculus for contravariant functors $F$ on $\mathcal{O}(P) \times \mathcal{O}(Q)$ (rather than on $\mathcal{O}(P \coprod Q)$). Get bitower

$$\begin{tikzcd}
T_{0,\infty}F(-,-) \arrow[d] & \cdots \arrow[l] & \cdots \arrow[l] & T_{\infty,\infty}F(-,-) \arrow[l] \arrow[d] \\
\vdots \arrow[d] & & & \vdots \arrow[d] \\
T_{0,1}F(-,-) \arrow[d] & T_{1,1}F(-,-) \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & T_{\infty,1}F(-,-) \arrow[l] \arrow[d] \\
T_{0,0}F(-,-) & T_{1,0}F(-,-) \arrow[l] & \cdots \arrow[l] & T_{\infty,0}F(-,-) \arrow[l]
\end{tikzcd}$$

Easy to see:

- $T_kF = \operatorname*{holim}_{k_1+k_2=k} T_{k_1,k_2}F$;
- Have same Goodwillie-Klein-Weiss convergence result.

# Construction of $T_{k_1,k_2}F$ for $F = \mathcal{LK}_2^d$

Let $I_1, \ldots, I_{k_1}$ be disjoint intervals in $\mathbb{R}$. Same for $J_1, \ldots, J_{k_2}$. Then

**Definition**

$$T_{k_1,k_2}\mathcal{LK}_2^d = \operatorname*{holim}_{\substack{\emptyset\neq S_1\subseteq\{1,\ldots,k_1\}\\ \emptyset\neq S_2\subseteq\{1,\ldots,k_2\}}} \mathrm{Emb}\left( (\mathbb{R}\setminus \bigcup_{i\in S_1} I_i) \coprod (\mathbb{R}\setminus \bigcup_{j\in S_j} J_j)\ ,\ \mathbb{R}^d \right).$$

Now get a bicosimplicial space $LK^{\bullet,\bullet}$ whose $(k_1, k_2)$ entry is $Conf(k_1 + k_2, \mathbb{R}^d)$.

**Proposition**

*For $d \geq 2$, $\mathrm{Tot}^{k_1,k_2} LK^{\bullet,\bullet} \simeq T_{k_1,k_2}\mathcal{LK}_2^d$.*

# Diagonal cosimplicial space and its $H^*$ spectral sequence

Now consider the diagonal cosimplicial space

$$LK^{\bullet,\bullet}_{diag} = \left\{ Conf(2k, \mathbb{R}^d) \right\}_{k=0}^{\infty}.$$

It is not hard to see that

$$\mathrm{Tot}\, LK_2^{\bullet,\bullet} = \mathrm{Tot}\, LK^{\bullet,\bullet}_{diag}.$$

Now have Bousfield-Kan cohomology spectral sequence for $LK^{\bullet,\bullet}_{diag}$ with

$$E_1^{-k,q} = H^q(Conf(2k, \mathbb{R}^d))$$

which converges to

$$\boxed{H^*(\mathrm{Tot}\, LK^{\bullet,\bullet}_{diag}) = H^*(\mathcal{L}\mathcal{K}_2^d), \quad d \geq 3.}$$

(This is a bit of a lie for $d = 3$.)

# What changes for homotopy string links and braids?

Again have bicosimplicial model for the bitower of punctured homotopy links, as well as its diagonal cosimplicial space, except:

- For $\mathcal{HLK}_2^d$, the $k$th space in the diagonal cosimplicial space is

  $$Conf(k,k;\mathbb{R}^d) = \{(x_1,\ldots,x_k,y_1,\ldots,y_k) \in (\mathbb{R}^d)^{2k} : x_i \neq y_j\}$$

  This is a kind of a "partial configuration space" or a complement of a hyperplane arrangement.
- For $\mathcal{BR}_2^d$, the $k$th space in the diagonal cosimplicial space is

  $$(Conf(2,\mathbb{R}^{d-1}))^k$$

  and this turns out to give the standard cosimplicial model for $\Omega Conf(2,\mathbb{R}^{d-1})$ (which is exactly what braids are).

# Generalization and main result

Generalization to $n$-component links is straightforward: Get

- $n$-dimensional Taylor towers for $\mathcal{LK}_n^d$, $\mathcal{HLK}_n^d$, and $\mathcal{BR}_n^d$;
- $n$-cosimplicial models for these towers;
- diagonal cosimplicial spaces $LK_{diag}^{\bullet,\bullet,\ldots,\bullet}$, $HLK_{diag}^{\bullet,\bullet,\ldots,\bullet}$, and $BR_{diag}^{\bullet,\bullet,\ldots,\bullet}$ modeling the towers;
- Bousfield-Kan cohomology spectral sequences for these cosimplicial spaces.

**Theorem (Munson, V.)**

*For $d = 3$, these spectral sequences collapse on the diagonal ($H^0$ part) at $E_2$.*

(This is true for $K^\bullet$ as well.) What is at this $E_2$?

# Real main result – Taylor towers and finite type invariants

Because cohomology of configuration spaces can be represented by chord diagrams, can see in $E_1$ a connection to *finite type knot and link invariants* (important because of connections to physics; conjectured to separate knots and links). In fact, not hard to see that $E_2$ contains precisely finite type invariants. Collapse then means

**Theorem (Main theorem revisited)**

*For* $X = \mathcal{K}^3, \mathcal{LK}^3_n, \mathcal{HLK}^3_n,$ *and* $\mathcal{BR}^3_n$,

$$H^0(T_{2k,2k,\ldots,2k}X) \cong \textit{ finite type } k \textit{ invariants.}$$

*(And* $H^0(T_{2k,2k,\ldots,2k}X) \cong H^0(T_{2k+1,2k+1,\ldots,2k+1}X)$*.)*

**Corollary**

*Recover Kontsevich's theorem about isomorphism between finite type invariants and (dual of) chord diagrams modulo four-term relation.*

# What next?

So the main result is

Taylor towers classify finite type invariants of knots and links.

To do:

- Reprove, in this setting, that finite type invariants separate braids (Kohno, Bar-Natan) and homotopy string links (Habegger-Lin).
- See if this helps in proving the same result for knots and links.
- Show rational collapse of Bousfield-kan spectral sequence for $\mathcal{LK}_n^d$, $\mathcal{HLK}_n^d$, and $\mathcal{BR}_n^d$ for $d > 3$. This would be the analog of the Lambrechts-Turchin-V. collapse result for $\mathcal{K}^d$ (collapse of the Vassiliev spectral sequence).
- Generalize Milnor invariants (which are finite type) to homotopy links of spheres in any dimension and connect with work of Koschorke. Show, using manifold calculus, that these generalizations suffice for separation of link maps of spheres.