11/12/07 ①

# Calculus of functors, operad formality, and embedding spaces

## Johns Hopkins topology seminar

Joint with (subsets of) G. Arone, P. Lambrechts, and V. Turchin

$M$ = smooth manifold
$V$ = vector space

Study rational homology of

$$\overline{\mathrm{Emb}}(M,V) := \mathrm{hofiber}\big(\underset{\substack{\text{space of}\\ \text{embeddings}\\ \text{of } M \text{ in } V}}{\underset{\shortparallel}{\mathrm{Emb}(M,V)}} \longrightarrow \underset{\substack{\text{space of}\\ \text{immersions}\\ \text{of } M \text{ in } V}}{\underset{\shortparallel}{\mathrm{Imm}(M,V)}}\big)$$

(note that we have to choose a basept. embedding $M \hookrightarrow V$)

or, study $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V)$, where

$H\mathbb{Q}$ = rat'l Eilenberg-MacLane spectrum

use
- embedding calculus of functors
- orthogonal calculus of functors
- Kontsevich's formality of little balls operad

# Embedding calculus (Goodwillie-Weiss)

$O(M)$ = category of open subsets of $M$ w/ inclusions
$O_k(M)$ = subcat. of $O(M)$, of open sets diffeomorphic to at most $k$ disjoint balls

If $F : O(M)^{op} \longrightarrow \begin{smallmatrix}\text{Top}\\\text{Spectra}\end{smallmatrix}$ is an

isotopy functor (plus another technical condition - i.e. takes filtered unions to homotopy colimits),

get tower

$$\begin{tikzcd}
 & T_\infty F(-) \arrow[d, dotted] & \\
 & T_{k+1} F(-) \arrow[d] & \overbrace{\Gamma_c\left(\binom{M}{k+1}, \widehat{F(k)}\right)}^{\text{we won't need this}} \arrow[l] \\
F(-) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & T_k F(-) := \operatorname{holim}_{U \in O_k(-)} F(U) \arrow[d, dashed] & \\
 & T_1 F(-) &
\end{tikzcd}$$

(Get a tower of functors and canonical maps as pictured. <u>Idea</u>: use <u>full functoriality</u> of $F$ to construct the tower and prove things about it, but then <u>evaluate it at $M = -$</u>)

(Under good circumstances, this tower <u>converges</u>, i.e. the map $F(-) \longrightarrow T_k F(-)$ gets better and better connected as you go up the tower. This is the sense in which $T_k F$ is "<u>$k^{th}$ Taylor polynomial of</u> $F$" and the tower is "<u>the Taylor tower</u>".)

(In particular, $F = \overline{Emb}(M, V)$ or
$F = H\mathbb{Q} \wedge \overline{Emb}(M, V)$
is one such functor.)

<u>Thm (Goodwillie–Weiss)</u> For $F = H\mathbb{Q} \wedge \overline{Emb}(M, V)$, the tower converges if $2\dim M + 1 < \dim V$

(and for $\overline{Emb}(M, V)$ if $\dim M + 2 < \dim V$)

(So strangely, <u>homology tower</u> is harder than the <u>homotopy tower</u>.)

(Think a little more about the stages of the embedding tower:)

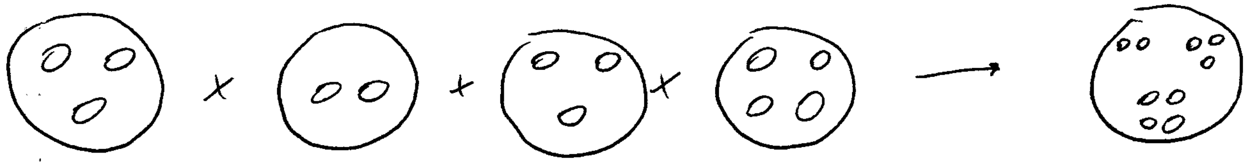
Let

$B(n,V)$ = Space of disjoint $n$-tuples of balls in unit ball of $V$.

and $B(\bullet,V)$ = little balls operad in $V$

( Have structure maps

$$B(k)\times B(\ell_1)\times \cdots \times B(\ell_k) \longrightarrow B(\Sigma \ell_i)$$



)

Then

$$T_k\, H\mathbb{Q}\wedge \overline{\mathrm{Emb}}(M,V) := \operatorname*{holim}_{U\in O_k(M)} H\mathbb{Q}\wedge \overline{\mathrm{Emb}}(U,V)$$

$$\text{“}\simeq\text{”} \operatorname*{holim}_{U\in O_k(M)} H\mathbb{Q}\wedge B(k_U, V)$$

some number $\leq k$ of balls in $V$

(Not really true as stated because $B(n,V)$ is not a functor of $O_k(M)$ and because there are various choices to be made, but this gives the right idea. Further, the maps in the diagram defining the stages $T_k$ are inclusions like the operad structure maps and holim imposes similar compatibility condition. So homology of $B(\bullet,V)$ might be helpful!)

(Actually, one can change the category $\mathcal{D}_2(n)$ a little and then "$\simeq$" becomes a real equivalence)

(But there is a really powerful theorem out there about the homology of the little disks operad:)

**Thm (Kontsevich):** $B(\bullet, V)$ is *formal*, i.e.

there are rational quasi-isomorphisms of CDGAs (commutative ~~[illegible]~~ differential graded algebras)

$$\big(C_*(B(n,V)), d\big) \longleftarrow \cdots \longrightarrow \big(H_*(B(n,V)), 0\big)$$

which respect the operad maps.

(This is really a remarkable theorem in rat'l homotopy theory. Kontsevich constructs a complex of certain chord diagrams in the middle which maps via integration to $C_*$ and has the right homology.)

Note: $H_*(B(n,V))$ = homology of configuration spaces $C(n,V)$ which is combinatorially easy; it can represented by chord diagrams modulo some relations

( and $H_*(B(\bullet,V))$ = Poisson operad; we'll see this again later )

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Thm (Arone–Lambrechts–V): If basept. embedding $M \hookrightarrow V$ factors through subspace $W \subset V$ such that $2 \dim W < \dim V$, then the diagram defining stages $T_k \overline{Emb}(M,V)$ is formal.

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( I.e. the functor $U \longmapsto \overline{Emb}(U,V)$, $U \in \mathcal{O}_k(M)$, is formal, so there are quasi-iso's of diagrams

$$C_* \overline{Emb}(U,V) \longleftarrow \cdots \longrightarrow H_* \overline{Emb}(U,V). \;)$$

( The assumption on factorization through a lower-dim'l subspace is there because at some point we need to consider inclusions $W \hookrightarrow V$ and we need a relative formality statement (due to Lambrechts–V)

$$\begin{array}{ccc} C_*(B(\bullet,W)) & \longrightarrow & C_*(B(\bullet,V)) \\ & \uparrow \vdots \downarrow & \\ H_*(B(\bullet,W)) & \longrightarrow & C_*(B(\bullet,V)) \end{array}$$

which requires this assumption. )

(So this is a statement about formality of chains. For a statement in terms of spectra, we have:)

<u>Cor:</u>

$$T_k\, H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \prod_{i=0}^{\infty} \underbrace{T_k}_{\substack{\| \\ \operatorname{holim} \\ U \in \mathcal{O}_k(M)}} \| \underbrace{H_i(B(k_U, V))}_{= H_i(C(k_U, V))} \|$$

where $\|H_i(X)\|$ = rat'l E-M spectrum whose $i^{th}$ homology is $H_i(X)$

(This comes from the fact that, if $F : \mathcal{C} \to$ Chain cplxs. is a formal functor, then

$$F \simeq \bigoplus F_n \quad \text{with} \quad H_*(F_n) = H_n(F_n),$$

and

$$\operatorname{holim} F \underset{\text{weq}}{\simeq} \prod \operatorname{holim}(H_n \circ F),$$

(since $H_* = \prod H_n$ and holim commutes with products) where $H_n$ : Ch. cplx $\to$ Ch. cplx assigns to a chain cplx the chain cplx w/ $H_n$ in $n^{th}$ slot and 0 differential.

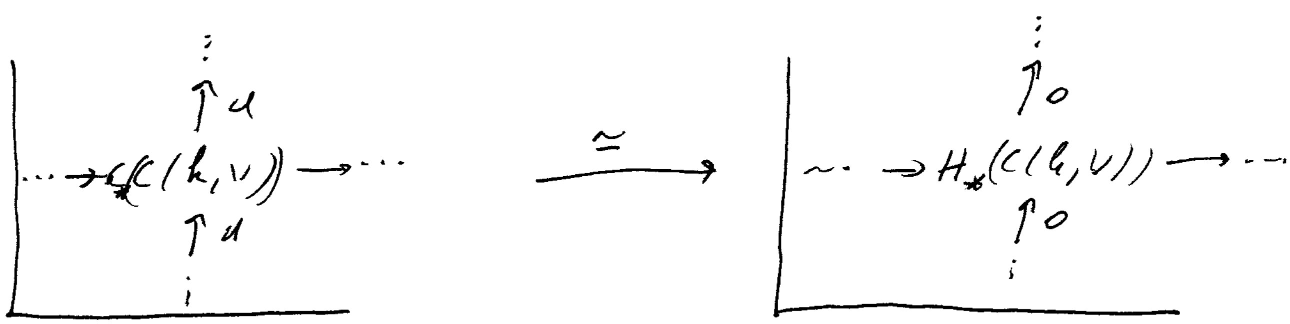
So have

$$T_k\, C_*\, \overline{\mathrm{Emb}}(M,V) = \prod T_k\, H_i\, \overline{\mathrm{Emb}}(M,V)$$

$$\Big\updownarrow \text{Quillen equivalence}$$

$$T_k\, H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) = \prod T_k \| H_i B(k_U, V) \|$$

(Now, what sometimes happens with formality is that one can deduce collapse of a spectral sequence (SS). Namely, suppose there is a filtration of some sort and this filtration is is in some sense compatible with the formality filtration – i.e. the trivial filtration by number of configuration points since formality is a statement in terms of a single configuration space – then one might hope for the rational collapse of the SS associated to this filtration since have




and so there is collapse at $E^2$.
However, the filtration in the embedding tower does not seem to lend itself nicely to this since each layer combines various ~~ ~~ configuration spaces in a non-trivial way.

But what <u>does</u> work is filtration associated to

<u>orthogonal calculus</u> )

## Orthogonal Calculus (Weiss)

$\mathcal{V} \equiv$ category of vector spaces with linear isometric inclusions

$F : \mathcal{V} \longrightarrow$ Top Spectra continuous

(i.e. $\mathrm{Map}(W, V) \times F(V) \longrightarrow F(W)$ continuous)

examples: $V \longmapsto BO(V),\ BU(V),\ S^V = V \cup \{\infty\},$
$\Omega^V S^V = \mathrm{maps}\left\{ \begin{matrix} V \to V \\ \infty \mapsto \infty \end{matrix} \right\},\ \underbrace{(H\mathbb{Q}\wedge)\mathrm{End}(M, V)}_{\text{the one we care about}}$

like in embedding calculus, get tower

$$
\begin{tikzcd}
 & P_\infty F(V) \arrow[d, dotted] & \\
 & \vdots \arrow[d] & \\
F(V) \arrow[ruu] \arrow[r] \arrow[rd] \arrow[rddd] & P_n F(V) \arrow[d] & D_n F(V) := \mathrm{hofiber}(P_n \longrightarrow P_{n-1}) \arrow[l] \\
 & P_{n-1} F(V) \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & P_1 F(V) &
\end{tikzcd}
$$

(Aside: $D_n \simeq \Omega^\infty(\underline{\underline{C}} \wedge S^{nV})_{hO(n)}$ where $S^{nV} = (\mathbb{R}^n \otimes V) \cup \infty$. This is a lot like the layers in homotopy calculus, and the two are in fact very closely related. In homotopy calculus, layers are $\Omega^\infty(\underline{\underline{C}} \wedge X^{\wedge n})_{h\Sigma_n}$ )

(We don't really need the description of the stages $P_n$ but the philosophy is that $F(V)$ should be studied for high values of $V$ and then extrapolate to low dimensions: stages are

$$P_n F(V) = \operatorname{hocolim}_i \left( T_n F(V) \to T_n(T_n F(V)) \to \cdots \to T_n^i F(V) \to \cdots \right)$$

where
$$T_n F(V) = \operatorname*{holim}_{U \subset \mathbb{R}^n} F(V \oplus U) \quad )$$

( Also, for $F = (H\mathbb{Q} \wedge)\overline{\mathrm{Emb}}(M, V)$, tower converges in the same range as embedding tower:

$$\begin{array}{ccc} P_n F & \xrightarrow[\text{large } k]{\simeq} & T_k P_n F \\ & & \simeq \uparrow \ \text{all } n, k \\ T_k F & \xrightarrow[\text{every } k]{\simeq} & P_n T_k F \end{array}$$

Taking inverse limit in $k$ or $n$ gives emb. or orth. tower. By above, inverse limit in either or both variables have to be same. Thus one converges iff the other does. )

So

$$\boxed{\text{Have same convergence for } (H\mathbb{Q}\wedge)\overline{\mathrm{Emb}}(M,V) \text{ as in embedding calculus tower}}$$

$$\boxed{\begin{array}{l}\text{Thm (Arone–Lambrechts–V): Suppose } 2\dim W < \dim V \\ \text{(same basept. embedding factorization condition). Then the} \\ \text{orthogonal tower for } H\mathbb{Q}\wedge \overline{Emb}(M,V) \text{ splits, i.e.} \\ \qquad P_n H\mathbb{Q}\wedge \overline{Emb}(M,V) \simeq \prod_{i=1}^{n} D_i H\mathbb{Q}\wedge \overline{Emb}(M,V)\end{array}}$$

(This is natural in $M$, but not necessarily in $V$)

Consequences:

1. Orthogonal tower gives rise to a spectral sequence which collapses at $E^1$. (For $\overline{Emb}(\mathbb{R}, \mathbb{R}^n)$, $n \geq 1$, this SS coincides w/ Vassiliev's.)

2. (It can be seen from an explicit description of $D_i$'s that they take rat'l homology equivalences to homotopy equivalences. We then get:)

   Suppose $f: M \to M'$ induces iso's on rational homology. Then if
   $$\dim V > 2\max(\text{Emb.Dim}(M), \text{Emb.Dim}(M')),$$
   $$\boxed{H_*(\overline{Emb}(M,V);\mathbb{Q}) \cong H_*(\overline{Emb}(M',V);\mathbb{Q})}$$

<u>Idea of pf</u>: Let $F(-) = H\mathbb{Q} \wedge \mathrm{Emb}(-, V)$

$$
\begin{aligned}
P_n F(M) &\overset{\text{large } k}{=\!=} T_k P_n F(M) \\
&= \operatorname*{holim}_{U \in \mathcal{O}_k(M)} P_n F(U) \\
&= \operatorname*{holim}_{U \in \mathcal{O}_k(M)} \prod_{i=1}^{n} D_i F(U) \\
&= \prod_{i=1}^{n} \operatorname*{holim}_{U \in \mathcal{O}_k(M)} D_i F(U) \\
&= \prod_{i=1}^{n} T_k D_i F(M) \\
&= \prod_{i=1}^{n} D_i F(M)
\end{aligned}
$$

This is because $F(M) \to T_k F(M)$ is an equivalence to about $k/2$. Applying $P_n$ preserves the equivalence so have $P_n F(M) \xrightarrow[\text{large } k]{\simeq} P_n T_k F(M)$, but we also have

$$P_n F(M) \xrightarrow{\simeq} T_k P_n F(M).$$

This is true because $T_k$ can be defined using a category whose nerve has finitely many nondegenerate simplices. (see 8.5 & 8.6 in paper).

(Formality implies that the orthogonal tower for $F(U) \simeq H\mathbb{Q} \wedge$ (configuration spaces) splits. More precisely

$$D_i(H\mathbb{Q} \wedge C(k,V)) \simeq \begin{cases} \| H_{i(\dim V - 1)} C(k,V) \|, & i \le k-1 \\ *, & \text{otherwise.} \end{cases}$$

<u>And</u>, functor

$$V \longmapsto \Sigma^\infty C(k,V) \text{ is polynomial}$$

of degree $k-1$

(Same step as first one, with $D_i$ replacing $P_n$)

Still to do:

1. Understand $E^1$ (we know what this is for knots, i.e. $\overline{Emb}(\mathbb{R}, \mathbb{R}^n)$, but this is a very special case, as explained next)

2. Homotopy version (we believe the homotopy SS also ~~[illegible]~~ collapses because of coformality of configuration spaces – $X$ coformal if $\exists$ quasi-iso's of ~~[illegible]~~ differential graded Lie algebras $\mathcal{L}(C_*(X)) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$. We have this collapse for knots! Ultimately we'd want to deduce that $\overline{Emb}(M, V)$ is a rational homotopy functor of $M$ and not just a rational homology functor.)

<u>Special case</u>: $\overline{\mathrm{Emb}}(\mathbb{R}, V)$ = long knots

(stages of the embedding tower simplify to homotopy limits of finite diagrams. In fact, one now gets a cosimplicial model for the tower, and we don't need to pass to the orthogonal tower.)

Let $C\langle k, V\rangle$ = Fulton–MacPherson-like compactification of $C(k, V)$ (collapse colinearities in F–M compactification)

and let $X^\bullet$ be the cosimplicial space

$$X^\bullet = \left\{ C\langle 0,V\rangle \rightrightarrows C\langle 1,V\rangle \underset{\leftleftarrows}{\overset{\to}{\rightrightarrows}} C\langle 2,V\rangle \underset{\overset{\leftleftarrows}{s^i}}{\overset{d^i}{\rightrightarrows}} \cdots \right\}$$

where $d^i$ = doubling
$s^i$ = forgetting

$$\boxed{\underline{\text{Thm}}\ (\underline{\text{Sinha}}):\quad \mathrm{Tot}^k X^\bullet \simeq T_k \overline{\mathrm{Emb}}(\mathbb{R}, V)}$$

($X^\bullet$ can also be thought of as the cosimplicial space associated to the Kontsevich operad (Sinha). This operad is $\simeq$ to little balls so $A$ is formal.)

$\boxed{\text{Thm (Lambrechts–Turchin–V): For } \dim V \geq 4, \text{ Operad formality} \Rightarrow \text{Bousfield–Kan } H_* \text{ SS for } X^\bullet \text{ collapses at } E^2 \text{ rationally}}$

$\boxed{\text{Thm (Arone–Lambrechts–Turchin–V): For } \dim V \geq 4, \text{ Operad coformality} \Rightarrow \text{Bousfield–Kan } \pi_* \text{ SS } (\otimes \mathbb{Q}) \text{ for } X^\bullet \text{ collapses at } E^2.}$

Consequences: ① ( $E^2$ of $H_*$ B-K SS can explicitly be identified w/ $E^1$ of Vassiliev's SS which he conjectured collapses and Kontsevich claimed he showed (but didn't)), so we have:)

$$\boxed{\text{Vassiliev's SS collapses at } E^1}$$

② $E^2$ of our SS has an easy combinatorial description; so we get: For $\dim V \geq 4$,

$$\boxed{H_* \overline{Emb}(\mathbb{R}, V) \cong HH_*(Poiss_{\dim V-1})},$$

where $Poiss_{\dim V-1}$ is the Poisson operad in dimension $\dim V-1$ and $HH_*$ is its Hochschild homology.

(.. Poisson operad of degree $n$; $k^{th}$ entry, is:

Start w/ $\underset{\text{symmetric algebra}}{\Lambda}\big(\underset{\text{Free Lie algebra}}{\mathcal{L}}(x_1,\dots,x_k)\big)$. Take submodule spanned by monomials in which all variables appear only once. Monomials are graded by putting all $x_i$ in degree 0 and giving the bracket degree $n$. So, for example, have elements

$$\underbrace{[x_1,x_3][[x_4,x_2],x_5]}_{\text{degree } 9}, \quad \underbrace{x_1x_2\cdots x_5}_{\text{degree } 0} \in Poiss_3(5)$$

Operad maps are induced by

$$\circ_i : Poiss_n(j)\otimes Poiss_n(k) \longrightarrow Poiss_n(j+k-1)$$
$$m_1 \otimes m_2 \longmapsto$$

- for each $j$, substitute $x_{j+i-1}$ for $x_j$ in $m_2$ to get $\bar{m}_2$.
- in $m_1$, substitute $m_2$ for $x_i$ and $x_{j+i-1}$ for $x_j$ if $j>i$ to get $m$.
- reduce $m$ according to graded Leibniz rule $[a,bc]=[a,b]c \pm b[a,c]$